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Quartic Spline Interpolation on Uniform Meshes with Application to Quadratures *

A.A.Karaballi, *S.Sallam*

(Dept. of Math. and Comp Sci., Kuwait University, P.O.Box 5969, Safat, 13060 Kuwait)

Abstract: In this paper an error estimate is derived, if $\lambda \in [0,1]$, for the lacunary quartic C^2-spline interpolant which interpolates the first derivative of a given smooth function at the mesh points and as well as its second derivative at an arbitrary interior point between consecutive knots, if the mesh is uniform. Moreover, an application to quadratures is also presented.

Key words: interpolation, quartic spline, quadrature formula.

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1. Introduction

Interpolation by quartic splines has been investigated by several authors (cf. [2], [3]). It was proved (see [4]) that given the real numbers: $\{f_i'\}_{i=0}^{N+1}$, $\{f_{i+\lambda}''\}_{i=0}^{N}$, and f_0, f_{N+1}, where $\lambda \in [0,1]$, then there exists a unique quartic spline $s \in S_{N,4}^{(2)}$ such that:

$$\begin{aligned} &s_i' = f_i', \ i = 0(1)N+1, \\ &s_{i+\lambda}'' = f_{i+\lambda}'', \ i = 0(1)N, \\ &s_0 = f_0, \ s_{N+1} = f_{N+1}, \end{aligned} \tag{1.1}$$

whenever $\lambda \neq \frac{1}{2} \pm \frac{1}{6}\sqrt{3}$, and N is even if $\lambda = \frac{1}{2}$. Also an L_∞-error estimate for this quartic spline, for only $\lambda = 0$ [resp. $\lambda = 1$] (cf. Theorem 2.p.357 [4]), was demonstrated.

The object of this paper is to extend the results of this theorem, i.e, the case $\lambda \in [0,1]$ will be considered. It is worth emphasizing here that we avoid using the inverse of the resulting tridiagonal matrix of the error vector $(e_1, e_2, \cdots, e_N)^T$ for $\lambda \in [0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ to obtain a bound for the error and that the error is not the best possible. We conclude with numerical test examples and a quadrature procedure to evaluate definite integrals.

2. Extension of the results and error estimation.

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We now proceed to formulate an extension to Theorem 2 (cf. [4]) for all $\lambda \in [0,1]$ as follows: Since the construction of $s \in S_{N,4}^{(2)}$ which satisfies (1.1) leads to the following linear system (see [3])

$$-(\lambda-1)(3\lambda-1)s_{i-1}+(1-2\lambda)s_i+\lambda(3\lambda-2)s_{i+1}=b_i, \tag{2.1}$$

where

$$b_i=(h/2)\left[(\lambda-1)(2\lambda-1)f'_{i-1}+(8\lambda^2-8\lambda+1)f'_i+\lambda(2\lambda-1)f'_{i+1}\right]+\left(h^2/12\right)\left(f''_{i-1+\lambda}-f''_{i+\lambda}\right),\ i=1(1)N,$$

and recalling that the ith component of $s(x)$ in $[x_i,x_{i+1}]$ is

$$s(x)=s_iA_\lambda(t)+s_{i+1}B_\lambda(t)+hf'_i\,C_\lambda(t)+hf'_{i+1}\,D_\lambda(t)+h^2f''_{i+\lambda}\,E_\lambda(t), \tag{2.2}$$

with

$$\begin{aligned}
A_\lambda(x)&=(x-1)^2(2x+1)-3(2\lambda-1)\mu x^2(1-x)^2,\\
B_\lambda(x)&=A_\nu(1-x),\\
C_\lambda(x)&=x(x-1)^2-(3\lambda-2)\mu x^2(1-x)^2,\\
D_\lambda(x)&=-C_\nu(1-x),\\
E_\lambda(x)&=\frac{1}{2}\mu x^2(1-x)^2,
\end{aligned}$$

and

$$\mu=(6\lambda^2-6\lambda+1)^{-1};\ \lambda\neq\frac{1}{2}\pm\frac{1}{6}\sqrt{3},\ \nu=1-\lambda,$$

then setting $e_i=s(x_i)-f(x_i)$, we have $(i=1(1)N)$,

$$-(\lambda-1)(3\lambda-1)e_{i-1}+(1-2\lambda)e_i+\lambda(3\lambda-2)e_{i+1}=d_i,$$

where

$$d_i=b_i+(\lambda-1)(3\lambda-1)f_{i-1}-(1-2\lambda)f_i-\lambda(3\lambda-2)f_{i+1}.$$

For the sake of brevity we will consider only the case where $l=3$, i.e., $f\in C^3[0,1]$ (the proof runs along the same lines for $l=5$).

Now let $z_i=e_{i+1}-e_i$, $i=0(1)N$, then (2.3) becomes

$$\beta z_{i-1}+z_i=\delta_i, \tag{2.4}$$

where

$$\beta=\frac{(\lambda-1)(3\lambda-1)}{\lambda(3\lambda-2)}\quad\text{and}\quad\delta_i=\frac{d_i}{\lambda(3\lambda-2)}.$$

Equation (2.4) is a first-order linear difference equation whose solution is

$$z_i=(-\beta)^i z_0+\sum_{j=1}^{i}(-\beta)^{i-j}\delta_j\ ,\ i=1(1)N.$$

Since $\sum_{i=1}^{N} z_i = -z_0$, then for $\lambda \in (\lambda_1, \frac{1}{2}) \cup (\lambda_2, 1]$, where λ_1 and λ_2 are the roots of the equation $6\lambda^2 - 6\lambda + 1 = 0$, we have

$$|z_0| \le \frac{1}{1-\beta}\left(|\delta_1| + \cdots + |\delta_N|\right),$$

i.e.,

$$|z_0| \le \frac{N\delta}{1-\beta} < \frac{d}{h\,|1-2\lambda|}, \tag{2.6}$$

where $\delta = \max\limits_{1\le i\le N} |\delta_i|$ and $d = \max\limits_{1\le i\le N} |d_i|$.

Taking the moduli of both sides of (2.5) and by the virtue of (2.6), we have

$$|z_i| \le |\beta|\,|z_0| + \delta \sum_{j=1}^{i} |\beta|^{i-j},$$

or

$$|z_i| \le |\beta|\,|z_0| + \frac{\delta}{1-|\beta|}, \quad i = 1(1)N.$$

It can be verified that:

$$1 - |\beta| = \begin{cases} \dfrac{1}{\mu\lambda(3\lambda-2)}, & \text{if } \lambda \in (\lambda_1, \frac{1}{3}] \cup (\lambda_2, 1], \\ \dfrac{2\lambda-1}{\lambda(3\lambda-2)}, & \text{if } \lambda \in [\frac{1}{3}, \frac{1}{2}), \end{cases}$$

consequently

$$|z_i| \le \begin{cases} \dfrac{|\beta|\,d}{h\,|1-2\lambda|} + |\mu|\,d, & \text{if } \lambda \in (\lambda_1, \frac{1}{3}] \cup (\lambda_2, 1], \\ \dfrac{\beta d}{h(1-2\lambda)} + \dfrac{d}{1-2\lambda}, & \text{if } \lambda \in [\frac{1}{3}, \frac{1}{2}), \end{cases}$$

with

$$d \le \begin{cases} \dfrac{3+7\lambda}{12} h^3 w_3(h), & \text{if } f \in C^3[0,1], \\ \dfrac{1}{720\,|\mu|} h^5 \left\| f^{(5)} \right\|_\infty, & \text{if } f \in C^5[0,1], \end{cases}$$

where $w_3(h)$ is the modulus of continuity of f''', [$w_3(h) \le 2\,\|f'''\|_\infty$] and this asserts the proof of the following lemma.

Lemma 2.1 *If $f \in C^3[0,1]$, then $(i = 0(1)N)$,*

$$|e_{i+1} - e_i| \le \begin{cases} \dfrac{3+7\lambda}{12\,|1-2\lambda|}\left[-\beta h^2 + \mu(2\lambda-1)\,h^3\right] w_3(h), & \text{if } \lambda \in (\lambda_1, \frac{1}{3}] \cup (\lambda_2, 1], \\ \dfrac{3+7\lambda}{12(1-2\lambda)}(\beta h^2 + h^3) w_3(h), & \text{if } \lambda \in [\frac{1}{3}, \frac{1}{2}). \end{cases}$$

Furthermore, if $f \in C^5[0,1]$, *then*

$$|e_{i+1}-e_i| \le \begin{cases} \dfrac{1}{720\mu(1-2\lambda)}\left[\beta h^4+\mu(1-2\lambda)h^5\right]\left\|f^{(5)}\right\|_\infty, & \text{if } \lambda\in(\lambda_1,\frac{1}{3}]\cup(\lambda_2,1], \\ \dfrac{1}{720\mu(2\lambda-1)}\left(\beta h^4+h^5\right)\left\|f^{(5)}\right\|_\infty, & \text{if } \lambda\in[\frac{1}{3},\frac{1}{2}). \end{cases}$$

Now we are in a position to prove the following theorem.

Theorem 2.1 *Let* $s(x)$ *be the quartic spline defined as above, with* $\lambda\in(\lambda_1,\frac{1}{2})\cup(\lambda_2,1]$. *If* $f\in C^3[0,1]$, *then for any* $x\in[0,1]$,

$$\left|s^{(r)}(x)-f^{(r)}(x)\right| \le \begin{cases} [C_\lambda+hg(\lambda)]h^{2-r}w_3(h), & r=1,2, \\ \frac{1}{2}[C_\lambda+hg(\lambda]h^2w_3(h), & r=0, \end{cases}$$

where

$$C_\lambda=\frac{\lambda(2-3\lambda)(3+7\lambda)\,|\mu\beta|}{1-2\lambda},$$

with

$$g(\lambda)=\begin{cases} \mu^2(3\lambda-2)(48\lambda^3-61\lambda^2+11\lambda-1), & \text{if } \lambda\in(\lambda_1,\frac{1}{3}], \\ \mu(27\lambda^3-69\lambda^2+39\lambda-4)(2\lambda-1)^{-1}, & \text{if } \lambda\in[\frac{1}{3},\frac{1}{2}), \\ \mu^2(3\lambda-2)(48\lambda^3-47\lambda^2+17\lambda-1), & \text{if } \lambda\in(\lambda_2,1]. \end{cases}$$

Proof Since $e_i=s(x_i)-f(x_i)$, then for $i=0(1)N$ and for $x\in[x_i,x_{i+1}]$, using Taylor's expansion of order 3, we have

$$h^2[s''(x)-f''(x)]=e_iA''_\lambda(t)+e_{i+1}B''_\lambda(t)+\Delta_{i,\lambda}(t),$$

where

$$\Delta_{i,\lambda}(t)=(\frac{h^3}{6})[(f^{(3)}(\alpha_1)-f^{(3)}(\alpha_2))B''_\lambda(t)+3\left(f^{(3)}(\alpha_3)-f^{(3)}(\alpha_4)\right)D''_\lambda(t)+ \\ 6t\left(f^{(3)}(\alpha_5)-f^{(3)}(\alpha_6)\right)];\ \alpha_i\in(x_i,x_{i+1}),\ i=1(1)6,$$

hence

$$|\Delta_{i,\lambda}(t)|\le\frac{h^3}{6}w_3(h)\left[\left|B''_\lambda(t)\right|+3\left|D''_\lambda(t)\right|+6\right].$$

In view of the validity of the following bounds

$$\left|B''_\lambda(t)\right|\le 12\mu\lambda(3\lambda-2),\ \text{if } \lambda\in(\lambda_1,\frac{1}{2})\cup(\lambda_2,1],$$

$$\left|D''_\lambda(t)\right|\le\begin{cases} 2\mu(12\lambda^2-9\lambda+1), & \text{if } \lambda\in(\lambda_1,\frac{1}{3}]\cup(\lambda_2,1], \\ 6\mu(4\lambda^2-5\lambda+1), & \text{if } \lambda\in[\frac{1}{3},\frac{1}{2}], \end{cases}$$

it follows that

$$|\Delta_{i,\lambda}(t)|\le\begin{cases} \mu(24\lambda^2-19\lambda+2)h^3w_3(h), & \text{if } \lambda\in(\lambda_1,\frac{1}{3}]\cup(\lambda_2,1], \\ \mu(24\lambda^2-25\lambda+4)h^3w_3(h), & \text{if } \lambda\in[\frac{1}{3},\frac{1}{2}]. \end{cases}$$

We now bound (2.10), using (2.11) and the results of Lemma (2.1) to arrive at

$$h^2 \left|s''(x) - f''(x)\right| \leq |z_i| \left|B''_\lambda(t)\right| + |\Delta_{i,\lambda}(t)|,$$

which obviously asserts the results given in (2.9) when $r = 2$.

Integrating over $[x_i, x]$,using $s'(x_i) = f'(x_i)$ and then a second time over $[0, x]$ (resp.over $[x, 1]$) if x is closer to 0 (resp. to 1), using $s(0) = f(0)$ (resp.$s(1) = f(1)$), inequalities (2.9) for $r = 1$ and $r = 0$ follow. This completes the proof .

Remark 1 *The case $\lambda = \frac{1}{2}$ (N even) will be briefly viewed as follows:*

If $\lambda = \dfrac{1}{2}$, then (2.3) can be written as $A\,\underline{e} = \underline{d}$, or, $\underline{e} = A^{-1}\underline{d}$ with $\left\|A^{-1}\right\|_\infty = \dfrac{N}{2}$, hence

$$|e_i| \leq \begin{cases} \frac{1}{6}h^2 w_3(h), & \text{if } f \in C^3[0,1], \\ \frac{1}{720}h^4 \left\|f^{(5)}\right\|_\infty, & \text{if } f \in C^5[0,1]. \end{cases}$$

Consequently for any $x \in [0, 1]$, we have

$$|s''(x) - f''(x)| \leq \begin{cases} (2 + 5h)w_3(h), & \text{if } f \in C^3[0,1], \\ \dfrac{1}{30}(1 + h)h^2 \left\|f^{(5)}\right\|_\infty, & \text{if } f \in C^5[0,1], \end{cases}$$

and

$$|s'(x) - f'(x)| \leq \begin{cases} \frac{1}{4}(2 + 3h)hw_3(h), & \text{if } f \in C^3[0,1], \\ \dfrac{1}{960}(4 + 7h)h^3 \left\|f^{(5)}\right\|_\infty, & \text{if } f \in C^5[0,1], \end{cases}$$

hence

$$|s(x) - f(x)| \leq \begin{cases} \frac{1}{8}(2 + 3h)h^2 w_3(h), & \text{if } f \in C^3[0,1], \\ \dfrac{1}{1920}(4 + 7h)h^4 \left\|f^{(5)}\right\|_\infty, & \text{if } f \in C^5[0,1]. \end{cases}$$

Also for the particular case $\lambda = \frac{1}{3}$, (2.1) will reduce to the backward recursive formula

$$s_i = s_{i+1} + \frac{h}{6}\left(2f'_{i-1} - 7f'_i - f'_{i+1}\right) + \frac{h^2}{4}\left(f''_{i-\frac{2}{3}} - f''_{i+\frac{1}{3}}\right),$$

with $s_{N+1} = f_{N+1}$ which computes s_i ($i = N(1)1$) and a similar argument can be used to obtain a bound for the error, for any $x \in [0, 1]$.

Remark 2 *It is worth noting that if we write (2.3) in the form $\gamma z_i + z_{i-1} = \sigma_i$, with*

$$\gamma = \frac{\lambda(3\lambda - 2)}{(\lambda - 1)(3\lambda - 1)} \quad \text{and} \quad \sigma_i = \frac{d_i}{(\lambda - 1)(3\lambda - 1)},$$

then for $\lambda \in [0, \lambda_1) \cup (\frac{1}{2}, \lambda_2)$ and using same arguments similar to that for the previous case we obtain similar results.

3. Application and Test Examples

In applied mathematics and statistics (cf. [1]) we sometimes confront with integrals of the type:

$$f(x) = \int_0^x f'(t)\mathrm{d}t,\ x \in [0,1].$$

This integrals can be evaluated by applying the procedure in Section 2, with fixed $\lambda \in [0,1]$, as follows:

- Use (2.1) to compute s_i, $i = 1(1)N$.
- Use (2.2) to compute $s(x)$.

Notice that in $[x_i, x_{i+1}]$ ($i = 0(1)N$) and $\lambda \neq 0$ or $1, s'(x)$ is a quasi-Hermite interpolant of f and one may look upon this as a competitive generalization to some traditional quadrature rules, which correspond to the cases $\lambda = 0$ (resp.$\lambda = 1$). One more advantage of this procedure is that the integral can be evaluated for any x with the same stepsize h.

In what follows we apply this procedure to the following test examples.

Example 1 Consider the function $f(x) = (x-1)(2x-1)\sin x$, in $[0,1]$.

Example 2 We test the proposed method on the well-known integral

$$f(x) = \frac{4}{\pi}\int_0^x \frac{\mathrm{d}t}{1+t^2}, \quad \text{in} \quad [0,1].$$

Example 3 We apply the procedure to evaluate the Fresnel integral

$$f(x) = \int_0^x \sin t^2 \mathrm{d}t, \quad \text{in} \quad [0,1].$$

In tables 1, 2 and 3 numerical results for error bounds are listed where the cases $N = 19$, 39,and 49, $h = 1/(N+1)$ with $\lambda = 0,\ 1/3,\ 1/2\ ,\ 2/3,$ and 1 are considered. The notation $e^{(j)}$; $j = 0,1,2$ stands for the maximum magnitude error $\left\|s^{(j)} - f^{(j)}\right\|_\infty$.

4. Conclusion

In summary, we have extended the results of [4] by deriving general error estimates, for all $\lambda \in [0,1]$, for the lacunary quartic C^2-spline that uses the first derivative of a smooth function at mesh points, and the second derivative at an arbitrary point between the knots.

It is worth noting that we avoided the usage of the coefficient matrix inverse of the error vector throughout our error analysis. In addition, the procedure can be implemented to evaluate definite integrals in a simple manner for any $x \in [0,1]$ without needing to change the stepsize.

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