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General Frames for Bivariate Interpolation *

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Abstract: Two classes of general bivariate interpolating frames are established by introducing multiple parameters. Many well known interpolating schemes, such as Newton interpolation, branched continued fraction interpolation proposed by Siemaszko and symmetric continued fraction interpolation considered by Cuyt and Murphy, can be obtained by choosing proper parameters in our results.

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1. Introduction

Let us briefly recall some classical results in univariate case. Given a set of real points $X=\{x_0,x_1,\cdots,x_n\}\subset[a,b]\subset\mathbb{R}$ and a function $f(x)$ defined in $[a,b]$, then one can construct a polynomial $P(x)$ of degree n

$$P(x)=a_0+a_1(x-x_0)+a_2(x-x_0)(x-x_1)+\cdots+a_n(x-x_0)\cdots(x-x_{n-1}), \tag{1.1}$$

where $a_i=f[x_0,x_1,\cdots,x_i]$ is the usual divided difference of the function $f(x)$ at points $x_0,x_1,\cdots,x_i$, such that it interpolates $f(x)$ on the set X. One may also establish a continued fraction of the following form ([5])

$$R(x)=b_0+\frac{x-x_0|}{|b_1}+\frac{x-x_1|}{|b_2}+\cdots+\frac{x-x_{n-1}|}{|b_n} \tag{1.2}$$

which interpolates $f(x)$ on the set X, provided that $b_i, i=0,1,\cdots,n$, are chosen to be the inverse differences of $f(x)$ at points $x_0,x_1,\cdots,x_i$. It is obvious that (1.1) and (1.2) can be rewritten as

$$\left\{\begin{array}{lcl} P_n(x) & = & a_n, \\ P_k(x) & = & a_k+(x-x_k)P_{k+1}(x),\ k=n-1,n-2,\cdots,0, \\ P(x) & = & P_0(x) \end{array}\right. \tag{1.3}$$

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and

$$\begin{cases} R_n(x) &= b_n, \\ R_k(x) &= b_k + (x - x_k)[R_{k+1}(x)]^{-1}, \ k = n-1, n-2, \cdots, 0, \\ R(x) &= R_0(x). \end{cases} \tag{1.4}$$

respectively. Now we may incorporate the above two schemes into a more general one

$$\begin{cases} S_n(x;\eta) &= u_n(\eta), \\ S_k(x;\eta) &= u_k(\eta) + (x - x_k)[S_{k+1}(x;\eta)]^{\eta}, \\ S(x;\eta) &= S_0(x;\eta), \end{cases} \tag{1.5}$$

where η takes the value 1 or -1, and $u_i(\eta) = F^{(\eta)}[x_0, x_1, \cdots, x_i]$ which is computed by the following steps

$$F^{(\eta)}[x_p] = f(x_p), \quad p = 0, 1, \cdots, n, \tag{1.6}$$

$$F^{(\eta)}[x_0, x_1, \cdots, x_i] = \left(\frac{F^{(\eta)}[x_0, x_1, \cdots, x_{i-2}, x_i] - F^{(\eta)}[x_0, x_1, \cdots, x_{i-1}]}{x_i - x_{i-1}} \right)^{\eta}, \tag{1.7}$$

Clearly we have

$$S(x;1) = P(x), \ \ S(x;-1) = R(x).$$

2. Bivariate schemes

Let $\Pi^{n,m}$ denote a set of real points in $\mathbb{R}^2$ which satisfies the inclusion property, i.e., if a point belongs to $\Pi^{n,m}$, then the rectangular subset of points emanating from the origin with the given point as its furthermost corner, also lies in $\Pi^{n,m}$. We may write $\Pi^{n,m}$ in the following form

$$\begin{aligned} \Pi^{n,m} &= \{(x_i, y_j) | j = 0, 1, \ldots, m_i; i = 0, 1, \ldots, n\} \\ &= \{(x_i, y_j) | i = 0, 1, \ldots, n_j; j = 0, 1, \ldots, m\}, \end{aligned} \tag{2.1}$$

where

$$m_0 = m, \ n_0 = n, \ m_0 \geq m_1 \geq \cdots \geq m_n; \ \ n_0 \geq n_1 \geq \cdots \geq n_m.$$

For $i = 0, 1, \ldots, n$, let

$$\begin{cases} s_{i,m_i}(y;\delta,\eta) &= a_{i,m_i}(\delta,\eta), \\ s_{i,k}(y;\delta,\eta) &= a_{i,k}(\delta,\eta) + (y - y_k)[s_{i,k+1}(y;\delta,\eta)]^{\delta}, \\ & \quad k = m_i - 1, \ldots, 1, 0, \\ s_i(y;\delta,\eta) &= s_{i,0}(y;\delta,\eta), \\ u_n(x,y;\delta,\eta) &= s_n(y;\delta,\eta), \\ u_i(x,y;\delta,\eta) &= s_i(y;\delta,\eta) + (x - x_i)[u_{i+1}(x,y;\delta,\eta)]^{\eta} \end{cases} \tag{2.2}$$

and for $j = 0, 1, \ldots, m$, let

$$\begin{cases} t_{n_j,j}(x;\delta,\eta) &= b_{n_j,j}(\delta,\eta), \\ t_{l,j}(x;\delta,\eta) &= b_{l,j}(\delta,\eta) + (x - x_l)[t_{l+1,j}(x;\delta,\eta)]^{\delta}, \\ & \quad l = n_j - 1, \ldots, 1, 0, \\ t_j(x;\delta,\eta) &= t_{0,j}(x;\delta,\eta), \\ v_m(x,y;\delta,\eta) &= t_m(x;\delta,\eta), \\ v_j(x,y;\delta,\eta) &= t_j(x;\delta,\eta) + (y - y_j)[v_{j+1}(x,y;\delta,\eta)]^{\eta}. \end{cases} \tag{2.3}$$

In the above defined recursive schemes, δ and η are chosen so that $|\delta| = |\eta| = 1$.

Theorem 2.1 *Given $f_{p,q}$ at each point $(x_p, y_q) \in \Pi^{n,m}$, let*

$$F^{(\delta,\eta)}[x_p; y_q] = f_{p,q}, \tag{2.4}$$

$$F^{(\delta,\eta)}[x_0,\ldots,x_i;y_q] = \left(\frac{F^{(\delta,\eta)}[x_0,\ldots,x_{i-2},x_i;y_q] - F^{(\delta,\eta)}[x_0,\ldots,x_{i-2},x_{i-1};y_q]}{x_i - x_{i-1}}\right)^{\eta}, \tag{2.5}$$

$$\begin{aligned} a_{i,j}(\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,\ldots,y_j] \\ &= \left(\frac{F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,\ldots,y_{j-2},y_j] - F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,\ldots,y_{j-2},y_{j-1}]}{y_j - y_{j-1}}\right)^{\eta}. \end{aligned} \tag{2.6}$$

Then $u_0(x,y;\delta,\eta)$ defined in (2.2) interpolates $f_{p,q}$ at point $(x_p,y_q) \in \Pi^{n,m}$.

Proof From (2.2) we know

$$\begin{aligned} s_{i,q-1}(y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,\ldots,y_{q-2},y_q], \\ s_{i,q-2}(y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,\ldots,y_{q-3},y_q], \\ \cdots \quad \cdots \quad \cdots & \quad \cdots \quad \cdots \quad \cdots \\ s_{i,1}(y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_i;y_0,y_q], \\ s_{i,0}(y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_i;y_q]; \end{aligned}$$

and

$$\begin{aligned} u_p(x_p,y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_p;y_q], \\ u_{p-1}(x_p,y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,\ldots,x_{p-2},x_p;y_q], \\ \cdots \quad \cdots \quad \cdots & \quad \cdots \quad \cdots \quad \cdots \\ u_1(x_p,y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_0,x_p;y_q], \\ u_0(x_p,y_q;\delta,\eta) &= F^{(\delta,\eta)}[x_p,y_q] = f_{p,q}. \end{aligned}$$

Theorem 2.1 is proved. Similarly one can prove the following

Theorem 2.2 *Given $g_{p,q}$ at each point $(x_p, y_q) \in \Pi^{n,m}$, let*

$$G^{(\delta,\eta)}[x_p; y_q] = g_{p,q}, \tag{2.7}$$

$$G^{(\delta,\eta)}[x_p; y_0, \dots, y_j] = \left(\frac{G^{(\delta,\eta)}[x_p; y_0, \dots, y_{j-2}, y_j] - G^{(\delta,\eta)}[x_p; y_0, \dots, y_{j-2}, y_{j-1}]}{y_j - y_{j-1}} \right)^{\eta}, \tag{2.8}$$

$$\begin{aligned} b_{i,j}(\delta,\eta) &= G^{(\delta,\eta)}[x_0, \dots, x_i; y_0, \dots, y_j] \\ &= \left(\frac{G^{(\delta,\eta)}[x_0, \dots, x_{i-2}, x_i; y_0, \dots, y_j] - G^{(\delta,\eta)}[x_0, \dots, x_{i-2}, x_{i-1}; y_0, \dots, y_j]}{x_i - x_{i-1}} \right)^{\eta}. \end{aligned} \tag{2.9}$$

Then $v_0(x, y; \delta, \eta)$ *defined in (2.3) interpolates* $g_{p,q}$ *at point* $(x_p, y_q) \in \Pi^{n,m}$.

As a matter of fact, (2.2) includes the following four different interpolating schemes.

Scheme 1

$$\begin{aligned} u_0(x, y; 1, 1) = s_0(y; 1, 1) + (x - x_0)s_1(y; 1, 1) + (x - x_0)(x - x_1)s_2(y; 1, 1) + \\ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})s_n(y; 1, 1), \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} s_i(y; 1, 1) = a_{i,0}(1, 1) + (y - y_0)a_{i,1}(1, 1) + (y - y_0)(y - y_1)a_{i,2}(1, 1) + \\ \cdots + (y - y_0)(y - y_1) \cdots (y - y_{m_i - 1})a_{i,m_i}(1, 1). \end{aligned} \tag{2.11}$$

Scheme 2

$$u_0(x, y; 1, -1) = s_0(y; 1, -1) + \frac{x - x_0 \rfloor}{\lfloor s_1(y; 1, -1)} + \frac{x - x_1 \rfloor}{\lfloor s_2(y; 1, -1)} + \cdots + \frac{x - x_{n-1} \rfloor}{\lfloor s_n(y; 1, -1)}, \tag{2.12}$$

where

$$\begin{aligned} s_i(y; 1, -1) = a_{i,0}(1, -1) + (y - y_0)a_{i,1}(1, -1) + (y - y_0)(y - y_1)a_{i,2}(1, -1) + \\ \cdots + (y - y_0)(y - y_1) \cdots (y - y_{m_i - 1})a_{i,m_i}(1, -1). \end{aligned} \tag{2.13}$$

Scheme 3

$$\begin{aligned} u_0(x, y; -1, 1) = s_0(y; -1, 1) + (x - x_0)s_1(y; -1, 1) + (x - x_0)(x - x_1)s_2(y; -1, 1) + \\ \cdots + (x - x_0)(x - x_1) \cdots (x - x_{n-1})s_n(y; -1, 1), \end{aligned} \tag{2.14}$$

where

$$s_i(y; -1, 1) = a_{i,0}(-1, 1) + \frac{y - y_0 \rfloor}{\lfloor a_{i,1}(-1, 1)} + \frac{y - y_1 \rfloor}{\lfloor a_{i,2}(-1, 1)} + \cdots + \frac{y - y_{m_i - 1} \rfloor}{\lfloor a_{i,m_i}(-1, 1)}. \tag{2.15}$$

Scheme 4

$$\begin{aligned} & u_0(x, y; -1, -1) \\ & = s_0(y; -1, -1) + \frac{x - x_0 \rfloor}{\lfloor s_1(y; -1, -1)} + \frac{x - x_1 \rfloor}{\lfloor s_2(y; -1, -1)} + \cdots + \frac{x - x_{n-1} \rfloor}{\lfloor s_n(y; -1, -1)}, \end{aligned} \tag{2.16}$$

where

$$s_i(y;-1,-1) = a_{i,0}(-1,-1) + \frac{y-y_0|}{|a_{i,1}(-1,-1)} + \frac{y-y_1|}{|a_{i,2}(-1,-1)} + \cdots + \frac{y-y_{m_i-1}|}{|a_{i,m_i}(-1,-1)}. \quad (2.17)$$

Therefore $u_0(x,y;1,1)$ is usual Newton interpolating polynomial and $u_0(x,y;-1,-1)$ stands for a kind of rational interpolation by branched continued fractions (see, e.g., [1,4]) while $u_0(x,y;1,-1)$ and $u_0(x,y;-1,1)$ represent blending interpolation schemes. Similarly the equation (2.3) also contains four different interpolating schemes, i.e., $v_0(x,y;1,1)$, $v_0(x,y;-1,-1)$, $v_0(x,y;1,-1)$ and $v_0(x,y;-1,1)$, and it is easy to see that $u_0(x,y;1,1) \equiv v_0(x,y;1,1)$ if $f_{p,q} = g_{p,q}$ for every $(x_p,y_q) \in \Pi^{n,m}$.

3. Symmetric schemes

Let

$$N = \max\{i|(x_i,y_i) \in \Pi^{n,m}\}, \quad (3.1)$$

$$\vec{\varepsilon} = (\varepsilon_0,\varepsilon_1,\cdots,\varepsilon_N), \quad \vec{\delta} = (\delta_0,\delta_1,\cdots,\delta_N), \quad (3.2)$$

where $|\varepsilon_i| = |\delta_i| = 1$ for $i = 0,1,\cdots,N$. With these symbols one can establish the following interpolation scheme

$$\begin{aligned}
g_{n_i,i}(x;\varepsilon_i) &= c_{n_i,i}, \\
g_{k,i}(x;\varepsilon_i) &= c_{k,i} + (x-x_k)[g_{k+1,i}(x;\varepsilon_i)]^{\varepsilon_i}, \quad k = i+1,\ldots,n_i-1, \\
g_i(x) &= g_{i+1,i}(x;\varepsilon_i), \\
h_{i,m_i}(y;\delta_i) &= c_{i,m_i}, \\
h_{i,l}(y;\delta_i) &= c_{i,l} + (y-y_l)[h_{i,l+1}(y;\delta_i)]^{\delta_i}, \quad l = i+1,\ldots,m_i-1, \\
h_i(y) &= h_{i,i+1}(y;\delta_i), \\
w_i(x,y;\varepsilon_i,\delta_i) &= c_{i,i} + (x-x_i)[g_i(x)]^{\varepsilon_i} + (y-y_i)[h_i(y)]^{\delta_i}, \quad i = 0,1,\ldots,N, \\
z_N(x,y;\vec{\varepsilon},\vec{\delta},\eta) &= w_N(x,y;\varepsilon_N,\delta_N), \\
z_i(x,y;\vec{\varepsilon},\vec{\delta},\eta) &= w_i(x,y;\varepsilon_i,\delta_i) + (x-x_i)(y-y_i)[z_{i+1}(x,y;\vec{\varepsilon},\vec{\delta},\eta)]^{\eta}, \quad i = 0,1,\ldots,N.
\end{aligned} \quad (3.3)$$

Theorem 3.1 *Given $h_{p,q}$ at each point $(x_p,y_q) \in \Pi^{n,m}$, let*

$$H[x_p;y_q] = h_{p,q}, \quad (3.4)$$

$$c_{i,i} = H[x_0,\ldots,x_i;y_0,\ldots,y_i]$$

$$= \left(\frac{\begin{array}{c} H[x_0,\ldots,x_{i-2},x_i;y_0,\ldots,y_{i-2},y_i] - H[x_0,\ldots,x_{i-1};y_0,\ldots,y_{i-2},y_i] \\ -H[x_0,\ldots,x_{i-2},x_i;y_0,\ldots,y_{i-1}] + H[x_0,\ldots,x_{i-1};y_0,\ldots,y_{i-1}] \end{array}}{(x_i-x_{i-1})(y_i-y_{i-1})} \right)^{\eta}, \quad (3.5)$$

$$c_{i,j} \quad (i<j) = H[x_0,\ldots,x_i;y_0,\ldots,y_j]$$

$$= \left(\frac{H[x_0,\ldots,x_i;y_0,\ldots,y_{j-2},y_j] - H[x_0,\ldots,x_i;y_0,\ldots,y_{j-1}]}{y_j - y_{j-1}} \right)^{\delta_i}, \quad (3.6)$$

$$c_{i,j} \quad (i > j) = H[x_0, \dots, x_i; y_0, \dots, y_j]$$
$$= \left(\frac{H[x_0, \dots, x_{i-2}, x_i; y_0, \dots, y_j] - H[x_0, \dots, x_{i-1}; y_0, \dots, y_j]}{x_i - x_{i-1}} \right)^{\varepsilon_j}. \tag{3.7}$$

Then $z_0(x, y; \vec{\varepsilon}, \vec{\delta}, \eta)$ defined in (3.3) interpolates $h_{p,q}$ at point $(x_p, y_q) \in \Pi^{n,m}$.

Proof We might as well suppose $p \leq q$ (the proof for the case when $p \geq q$ is similar). From (3.3) we know

$$\begin{aligned}
h_{p,q-1}(y_q; \delta_p) &= H[x_0, \dots, x_p; y_0, \dots, y_{q-2}, y_q], \\
h_{p,q-2}(y_q; \delta_p) &= H[x_0, \dots, x_p; y_0, \dots, y_{q-3}, y_q], \\
\dots \quad & \quad \dots \\
h_p(y_q) &= H[x_0, \dots, x_p; y_0, \dots, y_p, y_q],
\end{aligned}$$

therefore

$$z_p(x_p, y_q; \vec{\varepsilon}, \vec{\delta}, \eta) = w_p(x_p, y_q; \varepsilon_p, \delta_p) = H[x_0, \dots, x_p; y_0, \dots, y_{p-1}, y_q].$$

Since

$$(x_p - x_{p-1})[g_{p-1}(x_p)]^{\varepsilon_{p-1}} = H[x_0, \dots, x_{p-2}, x_p; y_0, \dots, y_{p-1}] - H[x_0, \dots, x_{p-1}; y_0, \dots, y_{p-1}]$$

and

$$h_{p-1}(y_q) = H[x_0, \dots, x_{p-1}; y_0, \dots, y_{p-1}, y_q],$$

we have

$$\begin{aligned}
& w_{p-1}(x_p, y_q; \varepsilon_{p-1}, \delta_{p-1}) \\
& \quad = H[x_0, \dots, x_{p-2}, x_p; y_0, \dots, y_{p-1}] + H[x_0, \dots, x_{p-1}; y_0, \dots, y_{p-2}, y_q] - \\
& \quad\quad H[x_0, \dots, x_{p-1}; y_0, \dots, y_{p-1}]
\end{aligned}$$

and hence

$$z_{p-1}(x_p, y_q; \vec{\varepsilon}, \vec{\delta}, \eta) = H[x_0, \dots, x_{p-2}, x_p; y_0, \dots, y_{p-2}, y_q].$$

Proceeding in much the same manner, one gets

$$z_1(x_p, y_q; \vec{\varepsilon}, \vec{\delta}, \eta) = H[x_0, x_p; y_0, y_q].$$

It follows from the equation (3.3) and the recursive expressions given in the theorem that

$$\begin{aligned}
g_{p,0}(x_p; \varepsilon_0) &= c_{p,0}, \\
g_{p-1,0}(x_p; \varepsilon_0) &= H[x_0, \dots, x_{p-2}, x_p; y_0], \\
\dots \quad & \quad \dots \\
g_0(x_p) &= g_{1,0}(x_p; \varepsilon_0) = H[x_0, x_p; y_0], \\
h_0(y_q) &= H[x_0; y_0, y_q],
\end{aligned}$$

which yields

$$w_0(x_p, y_q; \varepsilon_0, \delta_0) = H[x_p; y_0] + H[x_0; y_q] - H[x_0; y_0].$$

As a result, we finally obtain

$$z_0(x_p, y_q; \vec{\varepsilon}, \vec{\delta}, \eta) = H[x_p; y_q] = h_{p,q}.$$

Theorem 3.1 is proved.

If we choose $\eta = 1$ and $\varepsilon_i = \delta_i = 1$ for $i = 0, 1, \ldots, N$, then $z_0(x, y; \vec{\varepsilon}, \vec{\delta}, \eta)$ turns out to be the Newton interpolation polynomial over $\Pi^{n,m}$; if we choose $\eta = -1$ and $\varepsilon_i = \delta_i = -1$ for $i = 0, 1, \ldots, N$, then $z_0(x, y; \vec{\varepsilon}, \vec{\delta}, \eta)$ becomes the symmetric branched continued fraction discussed by Cuyt and Murphy et al (see [2,3]) which takes on the following form

$$H[x_0; y_0] + \sum_{k=1}^{n_0} \frac{x - x_{k-1}|}{|H[x_0, \cdots, x_k; y_0]} + \sum_{k=1}^{m_0} \frac{y - y_{k-1}|}{|H[x_0; y_0, \cdots, y_k]} +$$
$$\sum_{j=1}^{N} \frac{(x - x_{j-1})(y - y_{j-1})|}{|H[x_0, \cdots, x_j; y_0, \cdots, y_j] + \sum_{k=j+1}^{n_j} \frac{x - x_{k-1}|}{|H[x_0, \cdots, x_k; y_0, \cdots, y_j]} + \sum_{k=j+1}^{m_j} \frac{y - y_{k-1}|}{|H[x_0, \cdots, x_j; y_0, \cdots, y_k]}}.$$

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二元插值的一般框架

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摘 要: 本文通过引进多参数建立了二元插值的一般框架. 这样，许多著名的经典插值格式，如 Newton 插值、分叉连分式插值、对称连分式插值等均可视为本文的特殊情形.