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Multipliers of A^p and H^p Spaces*

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Abstract: In this paper, the author obtained some new results on the coefficient multipliers of A^p and H^p.

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1. Introduction

In this paper, all the functions we consider are supposed to be analytic in the open unit disk $D: |z| < 1$. Let

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p}, 0 < p < \infty;$$

$$M_\infty(r,f) = \max_{0 \le \theta < 2\pi} |f(re^{i\theta})|;$$

$$H^p = \{f : \|f\|_{H^p} = \sup_{0 \le r < 1} M_p(r,f) < \infty\}, 0 < p \le \infty;$$

$$G^p = \left\{ f : \|f\|_{G^p} = \left(\int_0^1 M_1^p(r,f) dr \right)^{1/p} < \infty \right\}, 0 < p < \infty;$$

and

$$A^p = \left\{ f : \|f\|_{A^p} = \left(\frac{1}{\pi} \iint_D |f(z)|^p dx\, dy \right)^{1/p} < \infty \right\}, 0 < p < \infty.$$

As it is well known, $f \in A^p$ if and only if $\int_0^1 M_p^p(r,f) dr < \infty$; moreover, H^p, G^p and A^p are all F-spaces. We denote by (A, B) the space of the multipliers from A to B. That is $(A,B) = \{g : f * g \in B$, whenever $f \in A\}$, where $f * g$ is Hadamard product of f and g. Let C denote a positive constant depending only on the indices $p, q, \ldots$. It may differ at different occurences even in the same formula. $[x]$ is the maximum integer not exceeding x.

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The coefficient multipliers of A^p and H^p into A^q, H^q and G^q ($0 < p \le 1 \le q < \infty$) have been studied in [1], [2] and [3]. In this paper, we give some corresponding results for $0 < p \le q \le 1$.

2 Multipliers of A^p

Lemma 2.1 *If $0 < p < \infty$, then $f \in A^p$ if and only if for any q and λ with $p \le q \le \lambda < \infty$,*

$$\int_0^1 (1 - r)^{\lambda\alpha - 1} M_q^\lambda (r, f)\, \mathrm{d}r < \infty, \tag{2.1}$$

where $\alpha = 2/p - 1/q$.

Proof If $f \in A^p$, then

$$\int_0^1 M_p^p (r, f)\, \mathrm{d}r < \infty, \tag{2.2}$$

and

$$M_p(r, f) \le C(1 - r)^{-1/p}, \quad M_\infty (r, f) \le C(1 - r)^{-2/p}.$$

Hence

$$\begin{aligned}\int_0^1 (1 - r)^{\lambda\alpha - 1} M_q^\lambda (r, f)\, \mathrm{d}r &= \int_0^1 (1 - r)^{\lambda\alpha - 1} \left(\frac{1}{2\pi} \int_0^{2\pi} \left|f(re^{i\theta})\right|^{q-p} \left|f(re^{i\theta})\right|^p \right)^{\lambda/q} \mathrm{d}r \\ &\le \int_0^1 (1 - r)^{\lambda\alpha - 1} [M_\infty (r, f)]^{(q-p)\lambda/q} [M_p^p (r, f)]^{\lambda/q - 1} M_p^p (r, f)\, \mathrm{d}r \\ &\le C \int_0^1 M_p^p (r, f)\, \mathrm{d}r < \infty.\end{aligned}$$

Conversely, if (2.1) holds, letting $\lambda = q = p$, we obtain (2.2). This implies $f \in A^p$.

Theorem 2.2 *Suppose $0 < p \le q \le 1$ and $m = [2/p]$. Then*

(1) $(A^p, G^q) = \{g : M_1(r, g^{(m)}) = O(1 - r)^{2/p - 1/q - m}\}$;

(2) $(A^p, A^q) = \{g : M_q(r, g^{(m)}) = O(1 - r)^{2/p - 1/q - m - 1}\}$.

Proof (1) Suppose $f \in A^p$, $g \in \{g : M_1(r, g^{(m)}) = O(1 - r)^{2/p - 1/q - m}\}$. Let $h = f * g$, then

$$h(\rho^2 z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho^2 e^{it}) g(z e^{-it})\, \mathrm{d}t, \qquad 0 < \rho < 1.$$

Differentiation with respect to z gives

$$\rho^{2m} h^{(m)}(\rho^2 z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho^2 e^{it}) g^{(m)}(z e^{-it}) e^{-imt}\, \mathrm{d}t$$

Hence

$$\rho^{2m} M_1(\rho^2 r, h^{(m)}) \le M_1(\rho^2, f) M_1(r, g^{(m)}) \le C(1 - \rho)^{1 - 1/q} M_q(\rho, f) M_1(r, g^{(m)}),$$

where Theorem 5.9 in [4] has been used. Taking $\rho = r$, we have

$$r^{2mq}(1 - r)^{(m-1)q} M_1^q(r^3, h^{(m)}) \le C(1 - r)^{q(2/p - 1/q) - 1} M_q^q(r, f).$$

It follows from Lemma 2.1 that $\int_0^1 (1-r)^{(m-1)q} M_1^q(r, h^{(m)})\,dr < \infty$. But by successive applications of Lemma of [5], this implies $h \in G^q$. Thus

$$\{g : M_1(r, g^{(m)}) = O(1-r)^{2/p-1/q-m}\} \subset (A^p, G^q).$$

Conversely, if $g \in (A^p, G^q)$, $f \in A^p$, then by the colsed graph theorem, $T: f \to f * g$ is a bounded linear operater. Let $f(z) = m!z^m(1-z)^{-m-1}$, and observe that

$$h(z) = f * g(z) = z^m g^{(m)}(z). \tag{2.3}$$

Now set $f_\rho(z) = f(\rho z)$, $h_\rho(z) = h(\rho z)$, where $0 < \rho < 1$. Obviously, $f_\rho \in A^p$, Furthermore, since $2/p < m + 1$, we have

$$M_p^p(r, f_\rho) \le C(1 - r\rho)^{1-(m+1)p},$$

and

$$\| f_\rho \|_{A^p} = \left(\int_0^1 2r M_p^p(r, f_\rho)\,dr \right)^{1/p} \le C(1-\rho)^{2/p-(m+1)}.$$

By the bounded property of T, we have

$$\| h_\rho \|_{G^q} \le \| T \| \, \| f_\rho \|_{A^p} \le C(1-\rho)^{2/q-(m+1)}.$$

But

$$\| h_\rho \|_{G^q} \ge \left(\int_\rho^1 M_1^q(r, h'_\rho)\,dr \right)^{1/q} \ge M_1(\rho, h'_\rho)(1-\rho)^{1/q}.$$

Therefore $M_1(\rho, h'_\rho) = O(1-\rho)^{2/p-1/q-m-1}$. It follows from Theorem 5.5 of [4] that $M_1(\rho^2, h) = M_1(\rho, h_\rho) = O(1-\rho)^{2/p-1/q-m}$. Setting $\rho^2 = r$ and combining with (2.3) we obtain $M_1(r, g^{(m)}) = O(1-r)^{2/p-1/q-m}$. Thus

$$(A^p, G^q) \subset \{g : M_1(r, g^{(m)}) = O(1-r)^{2/p-1/q-m}\}. \tag{2.4}$$

(2) Suppose $f \in A^p$, $g \in \{g : M_q(r, g^{(m)}) = O(1-r)^{2/p-1/q-m-1}\}$. If $h = f * g$, then

$$h(\rho^3 z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho^2 e^{it}) g(\rho z e^{-it})\,dt, \quad 0 < \rho < 1.$$

Differentiation with respect to z gives us that

$$\rho^{2m} h^{(m)}(\rho^3 z) = \frac{1}{2\pi} \int_0^{2\pi} f(\rho^2 e^{it}) g^{(m)}(\rho z e^{-it}) e^{-imt}\,dt.$$

Now set $\rho = r$ to conclude that

$$r^{2m} h^{(m)}(r^4 e^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} f(r^2 e^{it}) g^{(m)}(r^2 e^{i(\theta-t)}) e^{-imt}\,dt.$$

Let $G(z) = \sum_0^\infty \overline{g_n} e^{-i(n-m)\theta} z^n$, where g_n is Taylor coefficient of g. Obviously, for arbitrary $\theta \in [0, 2\pi]$, $G(z)$ is analytic in D. Hence, so is $G^{(m)}(z)$, and $\overline{G^{(m)}(r^2 e^{it})} = g^{(m)}(r^2 e^{i(\theta - t)})$. Let $F(z) = f(z) G^{(m)}(z)$. Then $F(z)$ is analtyic in D. By Theorem 5.9 in [4], we have

$$r^{2m} \left| h^{(m)}(r^4 e^{i\theta}) \right| \le M_1(r^2, F) \le C(1-r)^{1-1/q} M_q(r, F)$$

$$= C(1-r)^{1-1/q} \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(re^{it}) \right|^q \left| g^{(m)}(re^{i(\theta - t)}) \right|^q dt \right)^{1/q}.$$

Hence

$$r^{2mq} M_q^q(r^4, h^{(m)}) \le C(1-r)^{q-1} M_q^q(r, f) M_q^q(r, g^{(m)})$$

$$\le C(1-r)^{q(2/p-1/q)-1-mq} M_q^q(r, f).$$

It follows from Lemma 2.1 that $\int_0^1 (1-r)^{mq} M_q^q(r, g^{(m)}) dr < \infty$. But by successive applications of Lemma of [5], this implies $h \in A^q$.

Conversely, an argument similar to that used in proof of (2.4) now leads to that

$$(A^p, A^q) \subset \{g : M_q(r, g^{(m)}) = O(1-r)^{2/p-1/q-m-1}\}.$$

Corollary 2.3 *If* $0 < p \le q \le 1$, $f \in A^p$, *then its fractional integral* $f_{[\beta]} \in G^q$, *where* $\beta = 2/p - 1/q$.

Proof Let

$$g(z) = \sum_{n=1}^{\infty} \frac{n!}{\Gamma(n+1+\beta)} z^n.$$

Then

$$g(z) = \frac{1}{\Gamma(\beta)} \int_0^1 (1-\rho)^{\beta-1} (1-\rho z)^{-1} d\rho,$$

$$M_1(r, g^{(m)}) \le C \int_0^1 (1-\rho)^{\beta-1} (1-\rho r)^{-m} d\rho \le C(1-r)^{2/p-1/q-m}.$$

By Theorem 2.2(1), $f_{[\beta]} = f * g \in G^q$.

Theorem 2.4 *Suppose* $0 < p \le q \le 1$, $m = [2/p]$. *Then*

$$(A^p, H^q) = \{g : M_q(r, g^{(m)}) = O(1-r)^{2/p-m-1}\}.$$

Our proof will make use of the following lemma.

Lemma 2.5 *Suppose* $0 < q \le 1$. *If* $\int_0^1 (1-r)^{q-1} M_q^q(r, f') dr < \infty$, *then* $f \in H^q$.

Proof of Lemma 2.5 Without loss of generality, we may assume $f(0) = 0$. so that

$$f(z) = \int_0^1 f'(tz) z \, dt$$

Let $t_n = 1 - 2^{-n}, n = 0, 1, 2 \ldots$ Then

$$|f(re^{i\theta})| \leq \sum_{n=1}^{\infty} \int_{t_{n-1}}^{t_n} |f'(tre^{i\theta})| \mathrm{d}t \leq \sum_{n=1}^{\infty} 2^{-n} F(rt_n, \theta),$$

where $F(rt_n, \theta) = \max_{\rho \leq rt_n} |f'(\rho e^{i\theta})|$. By Theorem 32(2) of [6],

$$M_q^q(r, f) \leq C \sum_{n=1}^{\infty} 2^{-nq} M_q^q(rt_n, f').$$

But

$$\infty > \int_0^1 (1-t)^{q-1} M_q^q(t, f') \mathrm{d}t \geq \sum_{n=1}^{\infty} \int_{t_n}^{t_{n+1}} (1-t)^{q-1} M_q^q(rt, f') \mathrm{d}t$$

$$\geq q^{-1}(1-2^{-q}) \sum_{n=1}^{\infty} 2^{-nq} M_q^q(rt_n, f').$$

Thus $f \in H^q$.

Proof of Theorem 2.4 If $f \in A^p$, $g \in \{g: M_q(r, g^{(m)}) = O(1-r)^{2/p-m-1}\}$, $h = f * g$, then

$$r^{2mq} M_q^q(r^4, h^{(m)}) \leq C(1-r)^{q(2/p-1/q)-1-mq+1} M_q^q(r, f).$$

Hence

$$\int_0^1 (1-r)^{mq-1} M_q^q(r, h^{(m)}) \mathrm{d}r < \infty, \quad \int_0^1 (1-r)^{q-1} M_q^q(r, h') \mathrm{d}r < \infty.$$

It follows Lemma 2.5 that $h \in H^q$.

See the proof of (2.4) for the converse.

3 Multipliers of H^p

Theorem 3.1 *Suppose* $0 < p < q \leq 1, m = [1/p]$. *Then*

(1) $(H^p, H^q) = \{g: M_q(r, g^{(m)}) = O(1-r)^{1/p-m-1}\}$;

(2) $(H^p, A^q) = \{g: M_q(r, g^{(m)}) = O(1-r)^{1/p-1/q-m-1}\}$;

(3) $(H^p, G^q) = \{g: M_1(r, g^{(m)}) = O(1-r)^{1/p-1/q-m}$.

Proof (1) If $f \in H^p$, $g \in \{g: M_q(r, g^{(m)}) = O(1-r)^{1/p-m-1}\}$, $h = f * g$, then

$$r^{2mq}(1-r)^{mq-1} M_q^q(r^4, h^{(m)}) \leq C(1-r)^{q(1/p-1/q)-1} M_q^q(r, f).$$

It follows from Theorem 5.11 of [4] that $\int_0^1 (1-r)^{mq-1} M_q^q(r, h^{(m)}) \mathrm{d}r < \infty$. Hence $\int_0^1 (1-r)^{q-1} M_q^q(r, h') \mathrm{d}r < \infty$. This implies $h \in H^q$.

Conversely, by the method similar to that used in the proof of (2.4), we can prove the desired conclusion.

(2) Suppose $f \in H^p$, $g \in \{g: M_q(r, g^{(m)}) = O(1-r)^{1/p-1/q-m-1}\}$, $h = f * g$. Then

$$r^{2mq} M_q^q(r^4, h^{(m)}) \leq C(1-r)^{-mq+q(1/p-1/q)-1} M_q^q(r. f).$$

By Theorem 5.11 of [4], $\int(1-r)^{mq}M_q^q(r,h^{(m)})dr<\infty$. This impliers $h\in A^q$.

See the proof of (2.4) for the converse.

(3) An argument similar to that used in the proof of Theorem 2.2(1) leads to the desired conclusion.

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A^p 和 H^p 空间的乘子

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摘　要

给出了关于 A^p 和 H^p 空间系数乘子的一些结果