Journal of Mathematical Research & Exposition
Vol. 19, Supp., 171-176, April, 1999

Antisimple Radical of Hopf Module Algebras*

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Abstract: Let H be a Hopf algebra over a field. The antisimple H-radical $A_s(A)$ for a H-module algebra A is defined. $A_s(-)$ is shown to be a special H-radical and varies characterizations of antisimple H-module algebras are given.

Keywords: Hopf algebra, radical theory, special H-radical, antisimple H-radical

Classification: AMS(1991) 16N99, 16W30/CCL O153.3

Document code: A **Article ID**: 1000-341X(1999) supp-0171-06

1. Introduction and basic definitions

Radicals are an important tool in structure theory since they yield subdirect decomposition of the semisimple algebras. Recently, algebras with Hopf algebra actions become the subject of intense investigation (cd. [3] and [5]). So far, although some results of radicals of Hopf module algebras are introduced and investigated, the results on this topic for Hopf module algebras is rare. We know that not all the results of the theory of radicals of ordinary rings can be carried over to the theory of H-radicals. For example, A. V. Sidorov[5] gave an example to show that the ADS-theorem does not hold for H-radicals. In this paper, we will investigate the antisimple radical theory of Hopf module algebras.

Throughout this paper H denotes a Hopf algebra over a field K. That is, H is an algebra with 1, and a coalgebra over K with comultiplication $\Delta: H \to H \otimes H$ denotes for each $h \in H$, $\Delta(H) = \sum_{(h)} h_{(1)} \otimes h_{(2)}$, counit $\epsilon: H \to k$ and antipode $S: H \to H$.

For compeletness, we give the following rudiments of Hopf algebras. A is an H-module algebra if A is a K-algebra which is an H-module with H-module structure μ: $H \otimes A \to A$, written as $\mu(h \otimes a) = h \cdot a$, such that $h(ab) = \sum_{(h)} (h_{(1)} \cdot a)(h_{(2)} \cdot b)$, for all $a, b \in A$, $h \in H$, and $1_H a = a$, for all $a \in A$, where 1_H is the unit of H. The measuring is called unital if A has a

* **Received date**: 1995-09-05
Foundation item: Supported by the Natural Science Foundation of Shandong province (Q98A05113)
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a unit element 1 and if $h \cdot 1 = \epsilon(h)1$ for all $h \in H$. For examples, groups acting on algebras by automorphisms, group graded algebras, Lie algebras acting on algebras as derivations are all H-modules algebras, with $H = kG$, $H = (kG)^*$ and $H = U(L)$, the respective Hopf algebras

Let **H** be the category of all associative H-module algebras The objects of **H** are all associative H-module algebras The morphisms of **H** are those algebra homomorphisms $\varphi: A \to B$, $A, B \in$ **H**, which are also H-module maps Such a φ will be called an H-homomorphism. An ideal I of an H-module algebra A is called an H-ideal if the action of H on A leaves I invariant. If I is an H-ideal of A then A/I is an H-module algebra via $h(a + I) = (ha) + I$ for all $h \in H$, $a \in A$. It is easy to verify that H-ideals are the same as the kernel of H-homomorphisms

Throughout this paper, $I \trianglelefteq_H A$ means that I is an H-ideal of A and $A \to B$ will denote the fact that B is a H-homomorphic image of A. We refer to [6] for the basic notions and results of Hopf algebras and [3], [4], [5], [7] or [8] for radical theoretic terms

Definition *A class* **R** *of H-module algebras is called an H-radical class in the sense of Kursh-Amitsur, if R satisfies the following conditions*:

(1) **R** *is H-homomorphically closed*: $A \to B$ *and* $A \in$ **R** *imply* $B \in$ **R**;

(2) *every H-module algebra A contains a largest* **R**-*H-ideal* **R**(A) *called* **R**-*radical of A*;

(3) *for every* $A \in$ **H**, $R(A/\mathbf{R}(A)) = 0$.

For every H-radical class **R** the class $S = \{A \in \mathbf{H} \mid \mathbf{R}(A) = 0\}$ is called the semisimple class of **R** Clearly the radical class **R** consists of all H-module algebras A for which $\mathbf{R}(A) = A$. As is well-known, for any H-radical class **R**, $\mathbf{R} = \mathbf{USR}$. Moreover also $\mathbf{R}(A) = (A)\mathbf{S}$ holds for every H-module algebra A.

Definition *A class of H-module algebras* **M** *is*

(1) *H-regular if every nonzero H-ideal of* $A \in$ **M** *has a nonzero H-homomorphic image in* **M**.

(2) *hereditary if* $I \trianglelefteq_H A \in$ **M** *implies* $I \in$ **M**

Clearly, a hereditary class is always H-regular, but not conversely.

Let **R** and **R**$'$ be two H-radical classes, if for all H-module algebras A, $\mathbf{R}(A) \subseteq \mathbf{R}'(A)$, then denote it by $\mathbf{R} \subset \mathbf{R}'$. It is clear that $\mathbf{R} \subseteq \mathbf{R}'$ if and only if $\mathbf{SR}' \subseteq \mathbf{SR}$.

Let **N** be a class of H-module algebras We say that an H-radical class **R** is the upper H-radical class determined by **N** if **R** is the largest H-radical class for which all $A \in$ **N** are **R**-semisimple The upper H-radical determined by **N**, if exists, is denoted by **UN**. For some classes **N** the upper H-radicals **UN** may not exist, but if **M** is an H-regular class, **UM** exists

Definition *An H-module algebra A is called H-prime if the product of any two nonzero H-ideals is again nonzero. H-prime ideals are H-ideal with an H-prime factor algebra. Similarly, we can define H-semiprimeness.*

An H-ideal of a H-module algebra A is said to be an essential H-ideal of A (denoted by $I \trianglelefteq_H \circ A$), if for any $0 \neq K \trianglelefteq_H A$ we have $I \cap K \neq 0$.

Definition *A Class* $\mathbf{M}$ *of* H *-module algebras is called an* H *-special class, if* $\mathbf{M}$ *satisfies*

(S1) *Each* $A \in \mathbf{M}$ *is an* H *-prime* H *-module algebras.*

(S2) *If* $I \trianglelefteq_H A \in \mathbf{M}$ *implies* $I \in \mathbf{M}$.

(S3) *If* $I \trianglelefteq_H \cdot A$ *with* $I \in \mathbf{M}$ *implies* $A \in \mathbf{M}$.

If $\mathbf{M}$ *is an* H *-special class, then by* (S2), *we know that* $\mathbf{M}$ *is* H *-regular, thus the upper* H *-radical* $\mathbf{R} = \mathbf{U}\mathbf{M}$ *exists, we call* $\mathbf{R}$ *a special* H *-radical detimned by* $\mathbf{M}$.

We now give an intrinsic characterization for special H -radicals similar to that obtained by Gardner and Weigandt for rings [8].

Theorem 1 *If* $\mathbf{P}$ *is a hereditary* H *-radical class and* $\mathbf{P}$ *is the class of all* H *-prime* H *-module algebras, then* $\mathbf{SR} \cap \mathbf{P}$ *is always an* H *-special class. In fact it is the largest* H *-special class contained in* $\mathbf{SR}$. *If* $\mathbf{R}$ *is a special* H *-radial. Then* $\mathbf{R} = \mathbf{U}(\mathbf{SR} \cap \mathbf{P})$.

Proof It is trivial that $\mathbf{SR} \cap \mathbf{P}$ satisfies (S1) and (S2). Since $\mathbf{R}$ is hereditary, it is easy to prove that $\mathbf{SR}$ is closed under essential extensions. Thus $\mathbf{SR} \cap \mathbf{P}$ is also closed under essential extensions.

If $\mathbf{R}$ is a special H-radical, then there exists an H-special class $\mathbf{M}$ such that $\mathbf{R} = \mathbf{U}\mathbf{M}$ and hence $\mathbf{M} \subseteq \mathbf{SR}$. Also $\mathbf{M} \subseteq \mathbf{SR} \cap \mathbf{P}$ holds. Hence $\mathbf{R} = \mathbf{USR} \subseteq \mathbf{U}(\mathbf{SR} \cap \mathbf{P}) \subseteq \mathbf{U}\mathbf{M} = \mathbf{R}$. □

Definition *An* H *-module algebra* A *is said to be subdirectly irreducible (abbreviated as sdi) if the intersection of all nonzero* H *-ideals of* A *is not zero. We called* H *-module algebra* A *is psdi if it is an* H *-prime and sdi. We shall denote by* $H(A)$ *the heart of an sdi* H *-module algebra* A, *i.e.*, $H(A) = \cap\{I \trianglelefteq_H A \mid I \neq 0\}$.

In what follows, we will let $\mathbf{H}_D = \{A \in H \mid A$ *is a sdi* H *-module algebra*$\}$.

Proposition 2 *If* $A \in \mathbf{H}_D$, *then* $H(A)^2 = 0$ *or* $H(A) = H(A)^2$ *is a* H *-simple* H *-module algebra.*

Proposition 3 *Let* $\mathbf{R}$ *be a hereditary* H *-radical and* $A \in \mathbf{H}_D$. *Then* A *is* $\mathbf{R}$ *-semisimple if and only if* $H(A)$ *is* $\mathbf{R}$ *semisimple.*

Theorem 4 *The class* $\mathbf{M}$ *of all psdi* H *-module algebras is an* H *-special class.*

Proof Suppose that A is a psdi H -module algebra with heart $H(A)$ and $0 \neq I \trianglelefteq_H A$. For any $0 \neq J \trianglelefteq_H I$, by the primeness of A, we know $JI \neq 0$ and $IJI \neq 0$. Since IJI is an H -ideal of A, it follows that $H(A) \subseteq IJI \subseteq J$. Therefore, I is a psdi H -module algebra with heart $H(A)$. Thus $\mathbf{M}$ satisfies (S2).

Next we prove that $\mathbf{M}$ satisfies (S3). Suppose that $I \trianglelefteq_H \cdot A$ and $I \in \mathbf{M}$ with heart $H(I)$. We first prove that A is H -prime. For if I_1, I_2 are H -ideals of A such that $I_1 I_2 = 0$, then $(I_1 \cap I)(I_2 \cap I) = 0$. By the H -primeness of I, we have $I_1 \cap I = 0$ or $I_2 \cap I = 0$. Since $I \trianglelefteq_H \cdot A$, we get that $I_1 = 0$ or $I_2 = 0$, that is, A is H -prime. Secondly, we prove that A is sdi. Let K be an arbitrary nonzero H -ideal of A. Then $0 \neq I \cap K \trianglelefteq_H I$, hence $H(I) \subseteq I \cap K$ and $H(I) \subseteq K$. It follows that A is a psdi H -module algebra. This prove that $\mathbf{M}$ satisfies (S3).

Definition *The upper* H *-radical determined by the* H *-special class of all psdi* H *-module algebras is called antisimple* H *-rasical of* H *-module algebras and denote by* $A_s(-)$. *An* H *-module algebra* A *for which* $A_s(A) = A$ *is called antisimple.*

Analogously for rings, we have

Theorem 5 *For any H-module algebra A, $A_s(A) = \cap\{I \trianglelefteq_H A : A/I$ is a psdi H-module algebra$\}$.*

Proof For brevity, let $K = \cap\{I \trianglelefteq_H A : A/I$ is a psdi H-module algebra$\}$. It is clear that $A_s(A) \subseteq K$. We claim that K is A_s-radical. If K is not A_s-radical, then there exists an $I \trianglelefteq_H K$ such that K/I is psdi. Let $C = \{x \in A : xK \subseteq I\}$. Then by Lemma 5 in [5], it is easy to prove that $C \trianglelefteq_H A$ and C/I is the maximal H-ideal of A/I such that $(C/I) \cap (K/I) = 0$. Hence, A/C is an essential extension of $(C + K)/C$. Since $(C + K)/C \cong K/I$ and K/I is psdi, it follows that A/C is psdi. Then $C \supseteq K$ and $KK \subseteq I$, which contradicts K/I is psdi. Thus K is A_s-radical. This completes the proof.

The next result gives characterization of antisimple H-module algebras. Its proof is idential to those of the corresponding result for rings (see [7]).

Proposition 6 *The following are equivalent for an H-module algebra A.*

(a) *A is antisimple.*

(b) *Every H-homomorphic image of A is a subdirect sum of sdi H-modle algebras A_i $(i \in I)$ such that $H(A_i)^2 = 0$ for each $i \in I$.*

(c) *A does not contain any H-prime ideal P such that A/P has a minimal H-ideal.*

(d) *No H-ideal of A can be mapped H-homomorphically onto a nonzero simple H-module algebra.*

Proposition 7 *An H-module algebra A is antisimple if and only if for every H-homomorphic image $\bar{A}$ of A we have $(*)$: $\langle \bar{a} \rangle_H^2 \neq \langle \bar{a} \rangle_H$. For every nonzero principal H-ideal $\langle \bar{a} \rangle_H$ of $\bar{A}$.*

Proof Both antisimplicity and condition $(*)$ are obviously H-homomorphically invariant properties.

If A is an antisimple H-module algebra, for which $(*)$ does not hold, then there exists an H-homomorphic image $\bar{A} = A/I$ of A and $\bar{a} \in \bar{A}$ such that $\langle \bar{a} \rangle_H^2 = \langle \bar{a} \rangle_H$. Let $\Sigma = \{K \trianglelefteq_H A \mid a \notin K \supseteq I\}$. By Zorn's lemma, there exists a maximal H-riadial $K \in \Sigma$. Therefore A/K is an sdi H-module algebra with heart $(\langle a \rangle_H + k)/K = \langle a + K \rangle_H$ and $\langle a + K \rangle_H^2 = \langle a + K \rangle_H$. Therefore A is not an antisimple H-module algebra, a contradiction.

Conversely, if A is not antisimple, then A/I is sdi with idempotent heart K/I for some H-ideal I. Then we have $K/I = \langle \bar{a} \rangle_H$ and $\langle \bar{a} \rangle_H^2 = \langle \bar{a} \rangle_H$. The condition $(*)$ is not fulfilled.

Definition *Let $\mathbf{M}$ be an arbitrary class of H-module algebras.*

(a) *A non-H-simple H-module algebra A is called an $H_{\mathbf{M}}$-module algebra if*

(i) *$A/I \in \mathbf{M}$ for every nonzero H-ideal I of A.*

(ii) *Every minimal H-ideal of A belongs to $\mathbf{M}$.*

(b) *An H-simple H-module algebra A is an $H_{\mathbf{M}}$-module algebra if and only if $A \in \mathbf{M}$. The class of all $H_{\mathbf{M}}$-module algebras is denoted by $\mathbf{M}^*$ and we will assume that 0 belongs to every nonempty class of H-module algebras.*

The proof of the following two lemmas is straightforward.

Lemma 8 *For any hereditary class* $\mathbf{M}$ *of* H *-module algebras,* $\mathbf{M}^*$ *is* H *-homomorphically closed.*

Lemma 9 *Let* $\mathbf{M}$ *be a class of* H *-module algebras. If the class of all nilpotent* H *-module algebra* $\mathbf{N} \subseteq \mathbf{M}$, *then* $\mathbf{N} \subseteq \mathbf{M}^*$.

Lemma 10 *Let* $\mathbf{M}$ *be a clss of* H *-module algebras satisfying*

(a) $\mathbf{M}$ *is hereditary.*

(b) $\mathbf{M}$ *contains all nilpotent* H *-module algebras.*

(c) $\mathbf{M}$ *satisfies the extension property.*

Then $\mathbf{M}^*$ is hereditary, H -homomorphically closed and contains all the nilpotent H -module algebras

Proof By Lemma 8 and Lemma 9, it is clear that $\mathbf{M}^*$ is H -homomorphically closed and contains all nilpotent H -module algebras

To show that $\mathbf{M}^*$ is hereditary, let $A \in \mathbf{M}^*$ and $0 \neq I \triangleleft_H A$. If I is H -simple then I is a minimal H -ideal of A and hence $I \in \mathbf{M}$. Thus $I \in \mathbf{M}^*$. Suppose that I is not H -simple and K is any nonzero H -ideal of I. If $\langle K \rangle_H = I$, then I/K is nilpotent since $(\langle K \rangle_H)^3 \subseteq K$. By (b) of the hypothesis we have $I/K \in \mathbf{M}$. If $I = K$, it follows that $K \triangleleft_H A$ and we $A/K \in \mathbf{M}$. From the fact that M is hereditary and $I/K \triangleleft_H A/K \in A/K$, it follows that $I/K \in \mathbf{M}$. Let us therefore assume that $K \subseteq \langle K \rangle_H \subseteq I$. Then $(I/K)/(\langle K \rangle_H/K) \cong I/\langle K \rangle_H$ and $I/\langle K \rangle_H \in \mathbf{M}$ since it is an H -ideal of $A/\langle K \rangle_H \in \mathbf{M}$ and $\mathbf{M}$ is hereditary.

Furthermore $\langle K \rangle_H/K \in \mathbf{M}$ since it is nilpotent. And $I/K \in \mathbf{M}$ by (c).

If I contains a minimal H -ideal L, then $L^2 = 0$ or $L^2 = L$. By the definition of $\mathbf{M}^*$, $L \in \mathbf{M}$. □

Corollary 11 *If* $\mathbf{R}$ *is any supernilpotent* H *-radical class, then* $\mathbf{R} \subseteq \mathbf{R}^*$ *and* $\mathbf{R}^*$ *is hereditary and* H *-homomorphically closed.*

In order to give a characterization of antisimple H *-radical class* A_s *as a lower* H *-radical class we give*

Definition *A* H *-module algebra* A *is called an* H *-module algebra if every nonzero element* a *of* A *satisfies* $\langle a \rangle_H^2 \neq \langle a \rangle_H$.

If we denote the class of all a - H -module algebras by $\mathbf{K}$. We obtain

Theorem 12 $\mathbf{K}^* = A_s$.

Proof Let $A \in \mathbf{K}^*$. If A is H -simple, we have $A \in K$. Then, in view of the fact that $\langle a \rangle_H \neq (\langle a \rangle_H)^2$ for any $0 \neq a \in A$, it follows that A is a zero H -module algebra and hence $A \in A_s$. If A is a non H -simple H -module algebra, let A/I be any H -homomorphic image of A. Since $A \in \mathbf{K}^*$, we get $A/I \in \mathbf{K}$. For every $0 \neq \bar{a} \in \bar{A} = A/I$, it follows that $\langle \bar{a} \rangle_H^2 \neq \langle \bar{a} \rangle_H$. By Proposition 7 we have $\mathbf{K}^* \subseteq A_s$.

Conversely, if $A \in A_s$, the construction of $\mathbf{K}^*$ and Proposition 7 imply that $A \in \mathbf{K}^*$. Hence $A_s \subseteq \mathbf{K}^*$, so that the theorem is proved.

Acknowledgment The authors are greatful to Prof. E. R. Puczylowski and R. Wiegandt for sending their reprints of [4] and [8] respectively.

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Hopf 模代数的反单根

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摘 要

设H为域上Hopf代数. 本文定义了H-模代数A的反单H-根$A_s(A)$, 证明了$A_s(-)$为特殊H-根且给出反单H-模代数的各种刻画.