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The Grouth of Laplace-Stieltjes Transform Convergent Only in the Right Half Plane*

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Abstract On the analytic functions defined by exponential series convergent only in the right half plane, Yu Jiarong[1,2] introduced the order (R) and the order (R-H) and studied some exponential series, Yu Jiuman[3] introduced the proximate zero order (R) of the functions and obtained some results. For the study of the grouth of Laplace-Stieltjes transform, we introduce the order (R) and get some results similar to the case of exponential series.

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Let Laplace-Stieltjes transform

$$F(s) = \int_0^{\infty} e^{-sx} d\alpha(x), \tag{1}$$

where $s = \sigma + it$, σ and t being real variables, $\alpha(x)$ is of bounded variation on any interval $0 \le x \le b < +\infty$.

Take the sequences of real numbers

$$0 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots < \lambda_n \uparrow \infty, \tag{2}$$

satisfying the following conditions:

$$\overline{\lim_{n\to\infty}} \frac{n}{\lambda_n} < +\infty, \tag{3}$$

$$\overline{\lim_{n\to\infty}} (\lambda_{n+1} - \lambda_n) < +\infty, \tag{4}$$

In this paper we suppose that

$$\overline{\lim_{n\to\infty}} \log A_n^* / \lambda_n = 0 \tag{5}$$

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where $A_n^* = \sup_{\substack{\lambda_n < x \le \lambda_{n+1} \\ -\infty < t < +\infty}} \left| \int_{\lambda_n} e^{-ity} d\alpha(y) \right|$. Then the abscissa of absolute convergence of (1) is 0 and the transform (1) defines a function $F(S)$ analytic in the right half plane.

Let

$$M(\sigma, F) = \sup_{\substack{0 < x < +\infty \\ -\infty < t < +\infty}} \left| \int e^{-(\sigma + it)y} d\alpha(y) \right|,$$

$$m(\sigma, F) = \max_{1 \le n < +\infty} A_n^* e^{-\lambda_n \sigma}$$

$$\log^+ x = \begin{cases} \log x, & x \ge 1 \\ 0, & x < 1 \end{cases}$$

The quantity

$$\rho = \overline{\lim_{\sigma \to +0}} \frac{\log^+ \log^+ M(\sigma, F)}{\log \frac{1}{\sigma}}$$

is called order (R) of $F(S)$ in $\sigma > 0$, and if the limit

$$\rho = \lim_{\sigma \to +0} \frac{\log^+ \log^+ M(s, F)}{\log \frac{1}{\sigma}} \text{ exists,}$$

we call the function $F(S)$ that have order (R) ρ increasing orderly.

In order to study the order (R) of $F(S)$, we establish the following lemmas:

Lemma 1[1] *Let λ and P are positive constants, then $\varphi(\sigma) = \sigma^{-p} - \sigma\lambda (\sigma > 0)$, obtain minimum $(1 + p)\left(\frac{\lambda}{p}\right)^{\frac{p}{1+p}}$ in $\sigma = \left(\frac{p}{\lambda}\right)^{\frac{1}{p+1}}$.*

Lemma 2[1] *Let s and P are positive constants, and $0 < p < 1$, then $\varphi(x) = x^p - \sigma x \ (x \ge 0)$, obtain maximum $(1 - P)\left(\frac{p}{\sigma}\right)^{\frac{p}{1-p}}$ in $x = \left(\frac{p}{\sigma}\right)^{\frac{1}{1-p}}$.*

Lemma 3[3] *Let $\{a_n\}$ is sequence of complex numbers. C is a positive constant*

$$\overline{\lim_{n \to \infty}} [\log^+ |a_n| / \log \lambda_n] \le C.$$

If any $\beta > 0$ and $\gamma > 0$ $(0 < \beta < c)$, there would exists $Y > 0$ such that any y', y'' satisfying $\log y'' \ge (1 + \gamma) \log y'$ and $y' > Y$, exist n such that $y' \le \lambda_n \le y''$, and $|a_n| \ge \lambda_n^{c-\beta}$, then we would have increasing positive integers sequences $\{n_j\}$ such that

$$\lim_{j \to +\infty} \log^+ |a_{n_j}| / \log \lambda_{n_j} = c, \quad \lim_{j \to +\infty} \log \lambda_{j+1} / \log \lambda_j = 1.$$

Now we apply the lemmas to prove the following theorems

Theorem 1 *Suppose the transform (1) satisfies (2), (3), (4) and (5), then $F(S)$ is of order (R) $\rho \Leftrightarrow$*

$$\overline{\lim_{n \to \infty}} (\log^+ \log^+ A_n^*) / \log \lambda_n = \frac{\rho}{\rho + 1}, \tag{6}$$

where $\frac{\rho}{\rho + 1}$ must be replaced by 1 in the case $\rho = +\infty$.

Proof First we prove the necessity of the theorem in the case $0 < \rho < +\infty$.

Let $I(x; s+it) = \int e^{-(s+it)y} d\alpha(y)$, we have

$$\int_{\lambda_n} e^{-ity} d\alpha(y) = \int_{\lambda_n} e^{\sigma y} d_y I(y; \sigma+it)$$

$$= I(y; \sigma+it) e^{\sigma y} \Big|_{\lambda_n}^{x} - \sigma \int_{\lambda_n}^{x} e^{\sigma y} I(y; \sigma+it) \mathrm{d}y \ (x \geq \lambda_n).$$

Hence when $\sigma > 0$,

$$\left| \int_{\lambda_n} e^{-ity} \mathrm{d}\alpha(y) \right| \leq 2M(\sigma, F) e^{\sigma x}.$$

By (4), there exists a constant $\mu > 0$ such that $0 < \lambda_{n+1} - \lambda_n < \mu$.

Evidently when

$$\lambda_n \leq x \leq \lambda_{n+1}, \quad x \leq \lambda_n + \mu.$$

Hence we have

$$A_n^* \leq 2M(\sigma, F) e^{\sigma x} \leq 2M(\sigma, F) e^{(\lambda_n+\mu)\sigma}.$$

On the other hand, for any $\epsilon > 0$, there exists $\sigma_0 > 0$, when $0 < \sigma < \sigma_0$,

$$\log^+ M(\sigma, F) < \left(\frac{1}{\sigma}\right)^{\rho+\epsilon}.$$

Hence $\log^+ A_n^* \leq \log 2 + \left(\frac{1}{\sigma}\right)^{\rho+\epsilon} + (\lambda_n + \mu)\sigma$.

Take fixed $\sigma \in (0, \sigma_0)$, there exists an integer N, when $n > N$, $\lambda_n \sigma > \log 2$.

By Lemma 1, when $n > N$,

$$\log^+ A_n^* \leq (2\lambda_n + \mu)\sigma + \left(\frac{1}{\sigma}\right)^{\rho+\epsilon} \leq (1+\rho+\epsilon)\left(\frac{2\lambda_n+\mu}{\rho+\epsilon}\right)^{\frac{\rho+\epsilon}{1+\rho+\epsilon}}$$

Then

$$\overline{\lim_{n\to\infty}} \frac{\log^+ \log^+ A_n^*}{\log \lambda_n} \leq \overline{\lim_{n\to\infty}} \frac{\log(1+\rho+\epsilon) + \frac{\rho+\epsilon}{1+\rho+\epsilon}[\log(2\lambda_n+\mu) - \log(\rho+\epsilon)]}{\log \lambda_n}$$

$$= \frac{\rho+\epsilon}{1+\rho+\epsilon}.$$

It's proved that

$$\overline{\lim}_{n\to\infty} \frac{\log^+ \log^+ A_n^*}{\log \lambda_n} \leq \frac{\rho}{\rho+1}. \tag{7}$$

On the other hand, suppose $\overline{\lim}_{n\to\infty} (\log^+ \log^+ A_n^*)/\log \lambda_n < \frac{\rho}{\rho+1}$ then there exists $\rho' < \rho$ such that

$$\overline{\lim_{n\to\infty}} [(\log^+ \log^+ A_n^*)/\log \lambda_n] < \frac{\rho'}{\rho'+1},$$

hence there exsits a constant $C > 0$ such that

$$A_n^* < c\exp\left(\lambda_n^{\frac{\rho'}{1+\rho'}}\right) \quad (n = 1, 2, 3, \cdots).$$

Combining [4] and Lemma 2, when $\sigma > 0$ and $0 < \epsilon < \frac{1}{2}$,

$$M(\sigma, F) \le \sum_{n=0}^{\infty} A_n^* e^{-\lambda_n \sigma} \le c \sum_{n=0}^{\infty} \exp(\lambda_n^{\frac{\rho'}{1+\rho'}} - \lambda_n s)$$

$$\le c \sup_{n \ge 0} \exp[\lambda_n^{\frac{\rho'}{1+\rho'}} - \lambda_n(\sigma - \epsilon\sigma)] \sum_{n=0}^{\infty} e^{-\lambda_n \epsilon}$$

$$\le \exp[\frac{(\rho')^{\rho'}}{(1+\rho')^{1+\rho'}} \cdot (\frac{1}{(1-\epsilon)\sigma})^{\rho'}] o(\frac{1}{\epsilon\sigma}) \quad (\sigma \to 0^+).$$

Hence we have

$$\overline{\lim_{\sigma \to +0}} [\log^+ \log^+ M(\sigma, F)] / \log \frac{1}{\sigma} \le \rho' < \rho.$$

This is contradictory with the condition of the theorem. The necessity of this theorem is proved.

We can easily prove a sufficiency of the theorem similar to proof of the necessity.

Theorem 2 *Let transform (1) satisfies (2), (3), (4) and (5). Then function $F(S)$ that have order (R) ρ is reqularized increasing $\Leftrightarrow$*

(*i*) $\overline{\lim}_{n\to\infty} (\log^+ \log^+ A_n^*) / \log \lambda_n = \frac{\rho}{1+\rho}$,

(*ii*) *There exists a increasing positive integers sequences $\{n_j\}$ such that*

$$\lim_{j \to +\infty} (\log^+ \log^+ A_{n_j}^*) / \log \lambda_{n_j} = \frac{\rho}{1+\rho}, \quad \lim_{j \to \infty} \lambda_{n_{j+1}} / \lambda_{n_j} = 1.$$

Where $\frac{\rho}{\rho+1}$ must be replaced by 1 in the case $\rho = +\infty$.

Proof First we prove a sufficiency of the theorem in the case $0 < \rho < +\infty$.

For any $\epsilon > 0$ and sufficiently large j,

$$A_{n_j}^* e^{-\lambda_{n_j} \sigma} > \exp(\lambda_{n_j}^{\frac{\rho-\epsilon}{\rho+1-\epsilon}} - \lambda_{n_j} s).$$

Take σ_j satisfies

$$\lambda_{n_j} = [\frac{\rho - \epsilon}{(\rho + 1 - \epsilon)\sigma_j}]^{\rho+1-\epsilon},$$

when $\sigma_{j+1} < \sigma \le \sigma_j$, evidently have

$$1 = \lim_{j \to +\infty} (-\log \sigma_j + 1) / (-\log \sigma_j) = \lim_{j \to +\infty} (-\log \sigma) / (-\log \sigma_j),$$

$$M(\sigma, F) \ge \frac{1}{2} A_{n_j}^* e^{-\lambda_{n_j} \sigma} \ge \frac{1}{2} A_{n_j}^* e^{-\lambda_{n_j} \sigma_j} > \frac{1}{2} \exp(\lambda_{n_j}^{\frac{\rho-\epsilon}{1+\rho-\epsilon}} - \lambda_{n_j} \sigma_j),$$

By Lemma 2

$$M(\sigma, F) \ge \frac{1}{2} \exp[(1 - \frac{\rho - \epsilon}{1 + \rho - \epsilon}) (\frac{(\rho - \epsilon)/(1 + \rho - \epsilon)}{\sigma_j})^{\frac{(\rho-\epsilon)/(1+\rho-\epsilon)}{1-(\rho-\epsilon)/(1+\rho-\epsilon)}}]$$

$$= \frac{1}{2} \exp[(\frac{1}{1 + \rho - \epsilon}) (\frac{\rho - \epsilon}{(1 + \rho - \epsilon)} \sigma_j)^{\rho - \epsilon}].$$

Then

$$\varliminf_{\sigma\to+0}[\log^+\log^+ M(\sigma,F)/\log\frac{1}{\sigma}]$$
$$\geq \varliminf_{\sigma\to+0}\log^+\{\log\frac{1}{2}+[\frac{1}{1+\rho-\epsilon}(\frac{\rho-\epsilon}{(1+\rho-\epsilon)\sigma_j})^{\rho-\epsilon}]\}/\log\frac{1}{\sigma}$$
$$\geq \varliminf_{\sigma\to+0}\frac{\log^+\frac{1}{2}\left[\frac{1}{1+\rho-\epsilon}\left(\frac{\rho-\epsilon}{(1+\rho-\epsilon)\sigma_j}\right)^{\rho-\epsilon}\right]}{\log\frac{1}{\sigma}}$$
$$=\rho-\epsilon$$

Combining the above inequality and the theorem 1 we have

$$\lim_{\sigma\to+0}[\log^+\log^+ M(\sigma,F)/\log\frac{1}{\sigma}]=\rho.$$

Now we prove the necessity of the theorem.

By Theorem 1, i) evidently hold.

Hypotheses ii) is not hold, by Lemma 3 there exists $\beta(\rho>\beta)$ and γ such that

$$A_n^*<\exp(\lambda_n^{\frac{\rho-\beta}{1+\rho-\beta}})\quad n_j'\leq n\leq n_j'',$$

where n_j' and n_j'' are sufficiently large integers, $n_j''>n_{j+1}'$, $\log\lambda_{n_j''}>(1+\gamma)\log\lambda_{n_j'}(j=1,2,3,\cdots)$.

By Lemma 2

$$A_n^*e^{-\lambda_n\sigma}<\exp\{\frac{1}{1+\rho-\beta}[(\rho-\beta)/(1+\rho-\beta)s]^{\rho-\beta}\},\quad n_j'\leq n\leq n_j''. \tag{8}$$

Take $\epsilon>0$ such that

$$(\rho+\epsilon)/(1+\frac{\gamma}{2})<\rho-\eta,$$

where $0<\eta<\rho$. Then there exists a positive integer n_0, when $n>n_0$,

$$A_n^*e^{-\lambda_n\sigma}<\exp[\lambda_n^{\frac{\rho+\epsilon}{1+\rho+\epsilon}}-\lambda_n\sigma].$$

Let $n_j>n_0$, take s_j' such that

$$(\lambda_{n_j'})^{1+\frac{\gamma}{2}}=[(\rho+\epsilon)/(1+\rho+\epsilon)\sigma_j']^{1+\rho+\epsilon}.$$

By Lemma 2, for sufficiently large j, when $n_0\leq n\leq n_j'$,

$$\begin{aligned}A_n^*e^{-\lambda_n\sigma_j'}&\leq\exp[\frac{\rho+\epsilon}{1+\rho+\epsilon}\sigma_j']^{\frac{\rho+\epsilon}{1+\gamma/2}}[1-(\frac{\rho+\epsilon}{1+\rho+\epsilon}\sigma_j')^{\frac{1}{1+r/2}}\sigma_j']\}\\&<\exp[k(1/\sigma_j')^{\rho-\eta}],\end{aligned} \tag{9}$$

where k is a positive number.

For sufficiently large j, when $n \geq n_j^*$, take $\overline{\lambda}_{n''_j} = \lambda_{n'_j}^{1+\gamma} < \lambda_{n''_j}$ we have

$$A_n^* e^{-\lambda_n s'_j} \leq \exp[(\overline{\lambda}_{n''_j})^{\frac{\rho+\epsilon}{1+\rho+\epsilon}} - \overline{\lambda}_{n''_j}\sigma'_j]$$

$$= \exp\{[\frac{\rho+\epsilon}{1+\rho+\epsilon}s'_j]^{\frac{(\rho+\epsilon)(1+\gamma)}{1+\gamma/2}}[1-(\frac{\rho+\epsilon}{1+\rho+\epsilon}\sigma'_j)^{\frac{1+\gamma}{1+\gamma/2}}\sigma'_j]\}. \qquad (10)$$

Combining (8), (9) and (10), we see that for sufficiently large j

$$m(\sigma, F) \leq \exp[k_1(1/\sigma'_j)^{\rho-\eta_1}],$$

where k_1 is a positive number, $\eta_1 = \min\{\beta, \eta\}$.

Similar to the way proved Theorem 1, we can easily prove, for any sufficiently large j,

$$M(\sigma'_j, F) < \exp[k_2(1/\sigma'_j)^{\rho-\eta}],$$

where k_2 is a positive number.

This is contradictory with the condition of the theorem.

The proof is completed.

When $\rho = +\infty$, the proof is similar to the case $0 < \rho < +\infty$.

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在右半平面收敛的Laplace-Stieltjes 变换的增长性

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摘 要

本文研究了在右平面收敛的Laplace-Stieltjes 变换的增长性, 得到了与指数级数类似的结果