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A Note on the Paper "On Hua-Wang Type Inequalities"*

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Abstract In this paper, we point out a fault in Theorem B in [1], generalize Hua-Wang type inequalities given by Theorem A in [1], and prove them by using elementary mean value inequalities. It also makes the improved Theorem B its corollary.

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1. Introduction

Let δ, a be given positive numbers, $p \neq 0, 1$ be a real number,

$$\Omega = \{x \mid x_1 \geq 0, \cdots, x_n \geq 0, x_1 + \cdots + x_n \leq \delta\},$$

$$\Omega_1 = \{x \mid x_1 > 0, \cdots, x_n > 0, x_1 + \cdots + x_n \leq \delta\},$$

$$F_n(x) = (\delta - \sum_{i=1}^{n} x_i)^p + a^{p-1}\sum_{i=1}^{n} x_i^p, \tag{1}$$

$$k_n = a/(n+a), \quad h_n = 1/(n+a).$$

Paper [1] generalized the inequality

$$(\delta - x_1 - \cdots - x_n)^2 + a(x_1^2 + \cdots + x_n^2) \geq k_n\delta^2 (p = 2) \tag{2}$$

(the equality holds if and only if $x_1 = \cdots = x_n = h_n\delta$) given by Prof. L. K. Hua in [2] and the others given by Prof. Wang Chunglie in [3], and by using the majorization and dynamic programming gave the following two results (be called by a joint name Hau- Wang type inequalities):

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Theorem A *Let δ, a be given positive numbers, then for $p > 1 (p < 0)$, $x \in \Omega(\Omega_1)$ we have*

$$F_n(x) \geq k_n^{p-1}\delta \tag{3}$$

when $0 < p < 1$, the equality (3) reverses the direction. Under every circumstance, the equality holds if and only if $x_1 = \cdots = x_n = h_n\delta$.

Theorem B *Let δ be a given positive number, then for $p > 1$ $(p < 0)$, $x \in \Omega(\Omega_1)$ we have*

$$2^{1-p}\delta^p \leq (\delta - x_1 - \cdots - x_n)^p + n^{p-1}(x_1^p + \cdots + x_n^p) \leq n^{p-1}\delta^p; \tag{4}$$

when $0 < p < 1$, the equality (4) reverses the direction. Under every circumstance, the left equality holds if and only if $x_1 = \cdots = x_n = \delta/(2n)$; the right one holds if and only if

$$x = \Delta_i = (\overbrace{0, 0, \cdots, \delta}^{i}, 0, \cdots, 0) \quad (1 \leq i \leq n).$$

Here we first point out that for $p < 0$, the conclusion of Theorem B is not true. Because

$$\lim_{\substack{x_i \to +0 \\ x \in \Omega_1}} [(\delta - x_1 - \cdots - x_n)^p + n^{p-1}(x_1^p + \cdots + x_n^p)] = +\infty,$$

the second half of (4) is not true in Ω_1; $x = \Delta_i \notin \Omega_1$, the function has no meaning at Δ_i, so the equality can not hold at the point. The other conclusions of Theorem B are true except for this circumstance.

The main task we will deal with in this paper is to generalize the Theorem A. We will try to prove the generalized Theorem by avoiding the tools of majorization or dynamic programming but using elementary mean value inequalities. As a result, we make the improved Theorem B its corollary.

2 Generalization of Hua-Wang Type Inequalities

Lemma *Let $b_i \geq 0$, $\sum_{i=1}^{n} b_i = 1$, then*

(*I*) *when $p > 1$, we have*

$$n^{1-p} \leq \sum_{i=1}^{n} b_i^p \leq 1; \tag{5}$$

(*II*) *when $p < 0$, we have*

$$n^{1-p} \leq \sum_{i=1}^{n} b_i^p < +\infty \ (\text{here } b_i > 0); \tag{6}$$

(*III*) *when $0 < p < 1$, we have*

$$1 \leq \sum_{i=1}^{n} b_i^p \leq n^{1-p}. \tag{7}$$

The left equalities in (5), (6) and the right one in (7) hold if and only if $b_1 = \cdots = b_n = 1/n$; *the right equality in (5) and the left one in (7) hold if and only if* $b = \Delta_i(1) = (0, \cdots, 0, \overset{i}{1}, 0, \cdots, 0)$ $(1 \le i \le n)$; *when* $p < 0$, $\lim_{b_i \to +0} \sum_{i=1}^n b_i^p = +\infty$.

Proof Let $a = (a_1, \cdots, a_n)$ be a nonnegative sequence, $q = (q_1, \cdots, q_n)$ be a positive number weight sequence with $\sum_{i=1}^n q_i = 1$, let $\ne 0$ be a real number. Now let us consider the r-power weighted means of a

$$M_r(a, q) = \left(\sum_{i=1}^n q_i a_i^r\right)^{1/r}.$$

From [4], for each pair of numbers $r < s$, we have

$$M_r(a, q) \le M_s(a, q), \tag{8}$$

and the equality holds if and only if $a_1 = \cdots = a_n$. Hence

$$G(a, q) = M_0(a, q) \le M_1(a, q) = H(a, q), \tag{9}$$

where

$$G(a, q) = M_0(a, q) = \lim_{r \to 0} M_r(a, q) = \prod_{i=1}^n a_i^{q_i}$$

is geometric mean, and $H(a, q) = M_1(a, q) = \sum_{i=1}^n q_i a_i$ is algebraic mean.

Now when $p > 1$,

$$\frac{1}{n} = \sum_{i=1}^n \frac{1}{n} b_i = M_1(b, \{\frac{1}{n}\}) \le M_p(b, \{\frac{1}{n}\}) = \left(\sum_{i=1}^n \frac{1}{n} b_i^p\right)^{1/p},$$

we get

$$n^{1-p} \le \sum_{i=1}^n b_i^p \le \sum_{i=1}^n b_i = 1,$$

and the left equality holds if and only if $b_1 = \cdots = b_n = 1/n$, the right one holds if and only if $b = \Delta_i(1)$ $(1 \le i \le n)$. When $p < 0$, $-p > 0$, using (8), (9) we have

$$n = 1/\sum_{i=1}^n \frac{1}{n} b_i = 1/H(b, \{\frac{1}{n}\}) \le 1/G(b, \{\frac{1}{n}\})$$

$$= G(\{\frac{1}{b_i}\}, \{\frac{1}{n}\}) \le M_{-p}(\{\frac{1}{b_i}\}, \{\frac{1}{n}\}) = \left(\sum_{i=1}^n \frac{1}{n} b_i^p\right)^{-1/p},$$

then $n^{1-p} \le \sum_{i=1}^n b_i^p$, and the equality holds if and only if $b_1 = \cdots = b_n = 1/n$. Then we have inequality (6). Using the similar methods, we can prove the inequality (7). □

Theorem *Let* δ, *a be given positive numbers, then we have*

(*I*) *when* $p > 1$,

$$k_n^{p-1} \delta^p \le F_n(x) \le s^{p-1}(a) \delta^p, \quad x \in \Omega, \tag{10}$$

(II) when $p < 0$,

$$k_n^{p-1}\delta^p \le F_n(x) < +\infty,\ x \in \Omega_1, \tag{11}$$

(III) when $0 < p < 1$,

$$s^{p-1}(a)\delta^p \le F_n(x) \le k_n^{p-1}\delta^p,\quad x \in \Omega, \tag{12}$$

where

$$s(a) = \begin{cases} a, & a \ge 1, \\ 1, & 0 < a < 1. \end{cases}$$

The left equalities in (10), (11) and the right one in (12) hold if and only if $x_1 = \cdots = x_n = h_n\delta$, *the right equality in (10) and the left one in (12) hold if and only if* $x = \Delta_i(\delta) = (0, \cdots, 0, \overbrace{0}^{i}, 0, \cdots, 0)$ *for* $a \ge 1$ $(1 \le i \le n)$ *or* $x = (0, \cdots, 0)$ *for* $0 < a < 1$. *When* $p < 0$, $\lim_{\substack{x_i \to +0 \\ x \in \Omega_1}} F_n(x) = +\infty$.

Proof Each element x of $\Omega(\Omega_1)$ can be expressed as $x = tb$, $t \in [0, \delta]$ (relatively $t \in (0, \delta]$), where $b = (b_1, \cdots, b_n)$ is a direction factor with $b_i \ge 0$ (relatively $b_i > 0$) and $\sum_{i=1}^n b_i = 1$. In the direction b,

$$F_n(x) = F_n(tb) = f_b(t) = (\delta - t)^p + a^{p-1}\sum_{i=1}^n b_i^p t^p, \tag{13}$$

$$f_b'(t) = p[a^{p-1}\sum_{i=1}^n b_i^p t^{p-1} - (\delta - t)^{p-1}], \tag{14}$$

$$f_b'(t) = 0 \Leftrightarrow t = t(b) = \delta/[1 + a(\sum_{i=1}^n b_i^p)^{\frac{1}{p-1}}].$$

when $p > 1$ or $p < 0$,

$$\begin{cases} f_b'(t) > 0 \Leftrightarrow t > t(b), \\ f_b'(t) < 0 \Leftrightarrow t < t(b), \end{cases} \tag{15}$$

when $0 < p < 1$,

$$\begin{cases} f_b'(t) > 0 \Leftrightarrow t < t(b), \\ f_b'(t) < 0 \Leftrightarrow t > t(b), \end{cases} \tag{16}$$

When $p > 1$ or $p < 0$, in the direction b, $F_n(x) = f_b(t)$ has the minimum

$$g(b) = f_b(t(b)) = \delta^p[a(\sum_{i=1}^n b_i^p)^{\frac{1}{p-1}}/(1 + a(\sum_{i=1}^n b_i^p)^{\frac{1}{p-1}})]^{p-1},$$

which implies that

$$f_b(t) \ge f_b(t(b)) = g(b),\ t \in [0, \delta]\ (\textit{relatively}\ t \in (0, \delta]), \tag{17}$$

the equality holds if and only if $t = t(b)$. Because for $r \ne 0, -1$, the function

$$U(x) = [ax^{\frac{1}{r}}/(1 + ax^{\frac{1}{r}})]^r,\quad x > 0,$$

is strictly monotone increasing, using the inequality $n^{1-p} \leq \sum_{i=1}^{n} b_i^p$ in (5), (6) we have

$$g(b) \geq \delta^p [a(n^{1-p})^{\frac{1}{p-1}}/(1 + a(n^{1-p})^{\frac{1}{p-1}})]^{p-1} = k_n^{p-1}\delta^p, \tag{18}$$

the equality holds if and only if $b = b_0 = (1/n, \cdots, 1/n)$. Using (17), (18) we have for $p > 1$ ($p < 0$), $x \in \Omega(\Omega_1)$,

$$F_n(x) = f_b(t) \geq g(b) \geq g(b_0) = F_n(x_0) = k_n^{p-1}\delta^p, \tag{19}$$

where $x_0 = t(b_0)b_0 = (\delta/(n+a), \cdots, \delta/(n+a))$, the equality holds if and only if $x = x_0$ or $x_1 = \cdots = x_n = h_n\delta$. So the left inequalities in (10), (11) hold.

When $p > 1$, in the direction b, the maximum of $f_b(t)$ can only appear at some end of the interval $[0, \delta]$. If $a \geq 1$, using the right inequality $\sum_{i=1}^{n} b_i^p \leq 1$ in (5), we have

$$f_b(\delta) = a^{p-1}\sum_{i=1}^{n} b_i^p \delta^p \leq a^{p-1}\delta^p = s^{p-1}(a)\delta^p,$$

$$f_b(0) = \delta^p \leq a^{p-1}\delta^p = s^{p-1}(a)\delta^p,$$

which implies that

$$F_n(x) = f_b(t) \leq s^{p-1}(a)\delta^p, \quad x \in \Omega, \tag{20}$$

the equality holds if and only if $t = \delta$, $b = \Delta_i(1)$ or $x = \delta\Delta_i(1) = \Delta_i(\delta)$ $(1 \leq i \leq n)$. If $0 < a < 1$, then

$$f_b(\delta) = a^{p-1}\sum_{i=1}^{n} b_i^p \delta^p < \delta^p = s^{p-1}(a)\delta^p,$$

$$f_b(0) = \delta^p = s^{p-1}(a)\delta^p,$$

we also have the inequality

$$F_n(x) = f_b(t) \leq s^{p-1}(a)\delta^p, \quad x \in \Omega, \tag{21}$$

but now the equality holds if and only if $t = 0$ or $x = 0$, $b = (0, \cdots, 0)$. Now we have proved the inequality (10).

When $p < 0$, it is obvious that

$$\lim_{\substack{x_i \to +0 \\ x \in \Omega_1}} F_n(x) = +\infty. \tag{22}$$

From (19) and (22), the inequality (11) holds

When $0 < p < 1$, using the inequality (16) and (7) in the lemma, we can similarly prove the inequality (12). This complete the proof. □

Now that we have the comparatively general results, when $a = n$, we can get the improved Form of Theorem B:

Corollary 1 *Let $\delta > 0$, then the function*

$$G_n(x) = (\delta - \sum_{i=1}^{n} x_i)^p + n^{p-1}\sum_{i=1}^{n} x_i^p$$

satisfies

(I) *when* $p > 1$,

$$2^{1-p}\delta^p \le G_n(x) \le n^{p-1}\delta^p, \quad x \in \Omega, \tag{23}$$

(II) *when* $p < 0$,

$$2^{1-p}\delta^p \le G_n(x) < +\infty, \quad x \in \Omega_1, \tag{24}$$

(III) *when* $0 < p < 1$,

$$n^{p-1}\delta^p \le G_n(x) \le 2^{1-p}\delta^p, \quad x \in \Omega \tag{25}$$

The left equalities in (23), (24) *and the right one in* (25) *hold if and only if* $x_1 = \cdots = x_n = \delta/2n$, *the right equality in* (23) *and the left one in* (25) *hold if and only if* $x = \Delta_i(\delta)$ $(1 \le i \le n)$. *When* $p < 0$, $\lim_{\substack{x_i \to +0 \\ x \in \Omega_1}} G_n(x) = +\infty$.

Let $\delta, a > 0, p \ne 0, 1$ again, but

$$\Omega^* = \{x \,|\, x_i \ge 0, \sum_{i=1}^{\infty} x_i \le \delta\}, \quad \Omega_1^* = \{x \,|\, x_i > 0, \sum_{i=1}^{\infty}\} x_i \le \delta\}$$

$$F(x) = (\delta - \sum_{i=1}^{\infty} x_i)^p + a^{p-1}\sum_{i=1}^{\infty} x_i^p,$$

(where we assume that $\sum_{i=1}^{\infty} x_i^p$ is convergent.) From the Theorem that we got just now, let $n \to +\infty$, we have

Corollary 2 *Let* $\delta, a > 0$, *then*

(I) *when* $p > 1$,

$$0 < F(x) \le s^{p-1}(a)\delta^p, \quad x \in \Omega^*; \tag{26}$$

(II) *when* $p < 0$,

$$F(x) = +\infty, \quad x \in \Omega_1^*; \tag{27}$$

(III) *when* $0 < p < 1$,

$$s^{p-1}(a)\delta^p \le F(x) < +\infty, \quad x \in \Omega^*. \tag{28}$$

The right equality in (26) *and the left one in* (28) *hold if and only if* $x = \Delta_i^*(\delta) = (\overbrace{0, \cdots, 0, \delta}^{i}, 0, \cdots)$ $(1 \le i < +\infty)$. *Let* $x^n = (\overbrace{\delta/(n+a), \cdots, \delta/(n+a)}^{n}, 0, \cdots) \in \Omega^*$, *when* $p > 1$, $\lim_{n\to\infty} F(x^n) = 0$; *when* $0 < p < 1$, $\lim_{n\to\infty} F(x^n) = +\infty$.

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关于"华罗庚-王中烈型不等式"一文的注记

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摘 要

指出了[1]的定理2中的一个错误, 推广了[1]中定理1给出的华罗庚-王中烈型不等式, 避开控制不等式与动态规划模型等专门工具, 改用较为初等的平均值不等式证明之, 使改正后的[1]中的定理2成其推论