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BCC-Algebra and Integral Pomonoid*

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Abstract In this paper we investigate the relation between BCC-algebras and integral pomonoids. This is a generalization of a result by I. Fleischer in [2].

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An algebra $(X; *, 0)$ is called a BCC-algebra if it satisfies the following axioms:

(1) $((x * y) * (z * y)) * (x * z) = 0$;

(2) $x * x = 0$;

(3) $0 * x = 0$;

(4) $x * 0 = x$;

(5) $x * y = y * x = 0$ implies $x = y$.

The above definition is a dual form of the ordinary definition. Any BCK-algebra is a BCC-algebra, but there are BCC-algebras which are not BCK-algebras. A BCC-algebra is a BCK-algebra iff it satisfies the identity

(6) $(x * y) * z = (x * z) * y$

If $(X; *, 0)$ is a BCC-algebra, then the relation $\leq$ defined on X by

(7) $x \leq y$ iff $x * y = 0$

is a partial order on X with 0 as a smmallest element. Moreover, the relation has the following properties

(8) $(x * y) * (z * y) \leq x * z$;

(9) $x \leq y$ implies $x * z \leq y * z$ and $z * y \leq z * x$;

(10) $x * y \leq x$

A monoid is an algebra $(M; *, 1)$ with a binary operation $*$ and a nullary operation 1 (indentity) satisfying: for any x, y, z in M

(M1) $x * (y * z) = (x * y) * z$;

(M2) $x * 1 = 1 * x = x$

Suppose $\leq$ is a partial ordering on M such that for any x, y, z in M

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(PO) $y \leq z$ implies $x * y \leq x * z$ and $y * x \leq z * x$

Then $(M; \leq, *, 1)$ is said to be a partially ordered monoid (briefly, pomonoid).

If 1 is the greatest element of M on $\leq$, then we say that M is integral.

Given a BCC-algebra $(X; *, 0)$, for any a in X let a^{-1} be such a map from X to X that

$$x a^{-1} = x * a \quad \text{for all } x \text{ in } X.$$

Let $a^{-1} o b^{-1}$ denote the composition of the maps a^{-1} and b^{-1}, and denote

$$M(X) = \{a^{-1} o \cdots o b^{-1}: \{a, \cdots, b\} \textit{ is a finite subset in } X\}$$

We define a binary relation $\ll$ on $M(X)$ by

$$a^{-1} o \cdots o b^{-1} \ll u^{-1} o \cdots o v^{-1}$$

if and only if, for all x in X, we identically have $x a^{-1} o \cdots o b^{-1} \ll u^{-1} o \cdots o v^{-1}$ or equivalently, $(x a^{-1} o \cdots o b^{-1}) * (x u^{-1} o \cdots o v^{-1}) = 0$.

Theorem *Let $(X; *, 0)$ be a BCC- algebra. Then $(M(X); \ll, o, 0^{-1})$ is a an integral pomonoid.*

Proof Obviously, the operation o satisfies the associative law. Since for any x in X we have

$$x 0^{-1} o a^{-1} \cdots o b^{-1} = (x * 0) a^{-1} o \cdots o b^{-1} = x^{-1} o \cdots o b^{-1}$$

and

$$x a^{-1} o \cdots o b^{-1} o 0^{-1} = x a^{-1} o \cdots o b^{-1}$$

it follows that 0^{-1} is an identity of $M(X)$, hence $(M(X); o, 0^{-1})$ is a monoid.

Suppose $a_1^{-1} o \cdots o a_n^{-1} \ll b_1^{-1} o \cdots o b_m^{-1}$. Then for all x in X

$$x a_1^{-1} o \cdots o a_n^{-1} \leq x b_1^{-1} o \cdots o b_m^{-1}$$

and

$$(\cdots (x * a_1) * \cdots) * a_n \leq (\cdots (x * b_1) * \cdots) * b_m \qquad (*)$$

For any $u_1, \cdots, u_k$ in X, replace x by $(\cdots (x * u_1) * \cdots) * u_k$ we obtain

$$(\cdots (((\cdots (x * u_1) * \cdots) * u_k) * a_1) * \cdots) * a_n \leq (\cdots (((\cdots (x * u_1) * \cdots) * u_k) * b_1) * \cdots) * b_m$$

Rightly $*$ multiplying both sides of $(*)$ inegualit by u_1 we have

$$((\cdots (x * a_1) * \cdots) * a_n) * u_1 \leq ((\cdots (x * b_1) * \cdots) * b_m) * u_1$$

Repeating the above argument m times we obtain

$$(\cdots (((\cdots (x * a_1) * \cdots) * a_n) * u_1) * \cdots) * u_k \leq (\cdots (((\cdots (x * b_1) * \cdots) * b_m) * u_1) * \cdots) * u_k$$

Consequently

$$(u_1^{-1} o \cdots o u_k^{-1}) o (a_1^{-1} o \cdots o a_n^{-1}) \ll (u_1^{-1} o \cdots o u_k^{-1}) o (b_1^{-1} o \cdots o b_m^{-1})$$

and

$$(a_1^{-1} o \cdots o a_n^{-1}) o (u_1^{-1} o \cdots o u_k^{-1}) \ll (b_1^{-1} o \cdots o b_m^{-1}) o (u_1^{-1} o \cdots o u_k^{-1})$$

The fact that $\ll$ is a partical ordering on $M(X)$ follows from $\leq$ being a partial ording on X.

Since for any $a_1, \cdots, a_n$ in X

$$x a_1^{-1} o \cdots o a_n^{-1} = (\cdots (x * a_1) * \cdots) * a_n \leq (\cdots (x * a_1) * \cdots) * a_{n-1} \leq \cdots \leq x = x 0^{-1} \quad (by\ (10))$$

it follows that 0^{-1} is the greatest element of $M(X)$. Summarizing the above results we obtain that $(M(X); \lambda l, o, 0^{-1})$ is integral pomonoid.

The above results is a generalization of a result by I Fleischer in [2].

In BCK-algebra, $(M(X); \lambda l, o, 0^{-1})$ satisfies the commutativity.

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BCC-代数与整偏序幺半群

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摘　　要

本文研究BCC-代数与整偏序幺半群的关系, 所得结果是 I Fleischer 在[2]中相应结果的推广.