关于一类修正的三角插值多项式*

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摘 要: 本文构造出一个以$\{\theta_k=\frac{k}{n+1}\pi\}_{k=1}^{n}$为插值节点的$f(\theta)\in C_{2\pi}$且为奇函数的修正的三角插值多项式$W_n(f;r,\theta)$ (r为自然数). $W_n(f;r,\theta)$对每个以2π为周期的奇连续函数都能在全实轴上一致地收敛到$f(\theta)$; 若$f(\theta)\in C_{2\pi}^{j}(0\le j\le r-1)$且是奇的, $W_n(f;r,\theta)$对其收敛阶均达到最佳收敛阶.

关键词: 修正的三角插值多项式, 一致收敛, 最佳收敛阶.

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1 引 言

设$f(\theta)\in C_{2\pi}$且为奇函数, 取点组

$$\theta_k=\frac{k\pi}{N},\quad k=1,2,\cdots,n \tag{1.1}$$

其中$N=n+1$. 令

$$\begin{aligned}s_k(\theta)&=\frac{(\cos\theta-\cos\theta_1)\cdots(\cos\theta-\cos\theta_{k-1})(\cos\theta-\cos\theta_{k+1})\cdots(\cos\theta-\cos\theta_n)\sin\theta}{(\cos\theta_k-\cos\theta_1)\cdots(\cos\theta_k-\cos\theta_{k-1})(\cos\theta_k-\cos\theta_{k+1})\cdots(\cos\theta_k-\cos\theta_n)\sin\theta_k}\\&=(-1)^{k-1}\frac{\sin\theta_k\sin N\theta}{N(\cos\theta-\cos\theta_k)},\quad k=1,2,\cdots,n,\end{aligned} \tag{1.2}$$

$$S_n(\theta)=\sum_{k=1}^{n}f(\theta_k)s_k(\theta), \tag{1.3}$$

这样便得到不高于n阶的奇三角多项式. 对于(1.3)来说, 有

$$S_n(\theta_i)=f(\theta_i),\quad i=1,2,\cdots,n. \tag{1.4}$$

据Faber定理知道: $S_n(\theta)$并非对每个以2π为周期的奇连续函数$f(\theta)$都能在全实轴上一致地收敛. 为改善$S_n(\theta)$的收敛性和收敛阶, 本文仿文献[1]中的伯恩斯坦修正方法, 即放弃某些插值条件, 对(1.3)进行修正, 得到修正的奇插值多项式$W_n(f;r,\theta)$ (r为自然数). $W_n(f;r,\theta)$的具体构造如下:

令

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$$\Delta_h^r f(\theta)=\sum_{i=0}^{r}(-1)^{1+i}\binom{r}{i}f(\theta+ih), \tag{1.5}$$

$$\nabla_h^r f(\theta)=\sum_{i=0}^{r}(-1)^{1+i}\binom{r}{i}f(\theta-ih), \tag{1.6}$$

$$\tilde{\Delta}_h^r f(\theta)=\frac{1}{2}(\Delta_h^r f(\theta)+\nabla_h^r f(\theta)), \tag{1.7}$$

其中$\binom{r}{i}=\frac{r!}{i!(r-i)!}$, $h=\frac{\pi}{N}$, $N=n+1$. 对 $\tilde{\Delta}_h^r f(\theta)$ 有性质: 若 $f(\theta)$ 是奇函数, 则 $\tilde{\Delta}_h^r f(\theta)$ 也是奇函数. 同时还规定:

$$\theta_k-ih=\frac{(k-i)}{N}\pi=\theta_{k-i},\quad \theta_k+ih=\frac{(k+i)}{N}\pi=\theta_{k+i},$$

其中 $k=1,2,\cdots,n$, i 是任意自然数.

给定偶数$2l$(l 为自然数), 将节点组 $\theta_1<\theta_2<\cdots<\theta_n$ 按$2l$ 分成若干组, 不妨设分为 q 组, 即 $n=2lq+v$, $0\leq v<2l$, 在每一组中的第$2lt$ 点处让$W_n(f;r,\theta)$ 取值为A_{2lt}:

$$A_{2lt}=f(\theta_{2lt})+B_{2lt},\quad t=1,2,\cdots,q \tag{1.8}$$

其中$B_{2lt}=\frac{1}{2^r}\sum_{p=1}^{2l}(-1)^p\tilde{\Delta}_h^r f(\theta_{2l(t-1)+p})$. 于是$W_n(f;r,\theta)$ 可以表为

$$W_n(f;r,\theta)=\sum_{k=1}^{n}A_k s_k(\theta), \tag{1.9}$$

其中当 $k=2lt$, $t=1,2,\cdots,q$ 时,A_k 由(1.8) 给出, 余下的$A_k=f(\theta_k)$.

主要结果如下:

定理1 对任给的以2π为周期的奇连续函数 $f(\theta)$, 极限式$\lim_{n\to\infty}W_n(f;r,\theta)=f(\theta)$ 在全实轴上一致成立.

定理2 设 $f(\theta)\in C_{2\pi}^j$且为奇函数, $0\leq j\leq r-1$, 则

$$|W_n(f;r,\theta)-f(\theta)|=O\left(\frac{1}{n^j}\omega\left(f^{j},\frac{1}{n}\right)\right),$$

其中"O"与 θ, n, f, $\cdots$, $f^{(j)}$ 无关, $\omega(f^j,\delta)$ 为函数 $f^{(j)}(\theta)$ 的连续模.

记 $\lambda_n(\theta)$ 为Lebesgue 函数, 即

$$\lambda_n(\theta)=\sum_{k=1}^{n}|s_k(\theta)|.$$

定理3 设 $f(\theta)\in C_{2\pi}^r$且为奇函数, 则

$$|W_n(f;r,\theta)-f(\theta)|=O\left(\frac{1}{n^r}\omega\left(f^{r},\frac{1}{n}\right)(1+\lambda_n(\theta))\right)$$

其中"O"与 θ, n, f, $\cdots$, $f^{(r)}$ 无关.

2 引 理

引理 成立如下估计式:

$$\sum_{k=1+r}^{n}\frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}\left|(-1)^{1+i}s_{k-i}(\theta)+s_k(\theta)\right|=O(1), \tag{2.1}$$

$$\sum_{k=1}^{n-r}\frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}\left|(-1)^{1+i}s_{k+i}(\theta)+s_k(\theta)\right|=O(1),\tag{2.2}$$

$$\sum_{t=1}^{q}\sum_{p=1}^{l-1}\left|s_{2lt}(\theta)-s_{2l(t-1)+2p}(\theta)\right|=O(1),\tag{2.3}$$

$$\sum_{t=1}^{q}\sum_{p=1}^{l}\left|s_{2lt}(\theta)+s_{2l(t-1)+2p-1}(\theta)\right|=O(1).\tag{2.4}$$

证明 由于(2.1), (2.2), (2.3)和(2.4)的证明方法是相同的, 故这里只证(2.4). 记 $i=2lt$, $j=2l(t-1)+2p-1$, 那么$1\le i-j=2(l-p)+1\le 2l-1$. 由

$$\frac{2\sin\theta_k}{\cos\theta-\cos\theta_k}=\cot\frac{\theta_k-\theta}{2}+\cot\frac{\theta_k+\theta}{2},\tag{2.5}$$

$$\left|\sin\frac{\theta_k-\theta}{2}\right|\le\left|\sin\frac{\theta_k+\theta}{2}\right|,\tag{2.6}$$

及(1.2), 有

$$\begin{aligned}\left|s_i(\theta)+s_j(\theta)\right| &= \left|\frac{\sin N\theta}{2N}\left(\frac{2\sin\theta_j}{\cos\theta-\cos\theta_j}-\frac{2\sin\theta_i}{\cos\theta-\cos\theta_i}\right)\right| \\ &= \left|\frac{\sin N\theta}{2N}\sin\frac{\theta_i-\theta_j}{2}\left(\frac{1}{\sin\frac{\theta_i-\theta}{2}\sin\frac{\theta_j-\theta}{2}}+\frac{1}{\sin\frac{\theta_i+\theta}{2}\sin\frac{\theta_j+\theta}{2}}\right)\right| \\ &\le \left|\frac{\sin N\theta}{N}\sin\frac{\theta_i-\theta_j}{2}\frac{1}{\sin\frac{\theta_i-\theta}{2}\sin\frac{\theta_j-\theta}{2}}\right|.\end{aligned}\tag{2.7}$$

先令$0\le\theta\le\pi$, 对于给定的 θ, 在$\{\theta_{2lt}\}_{t=1}^{q}$中, 设 θ_{2lt_0}离 θ最近, 同时又不妨设 $\theta_{2lt_0}\le\theta$(对于 $\theta>\theta_{2lt_0}$时的证法一致), 使用(2.7)和

$$\frac{2}{\pi}\theta\le\sin\theta\le\theta,\quad 0\le\theta\le\frac{\pi}{2},\tag{2.8}$$

$$s_k(\theta)=O(1),\quad k=1,2,\cdots,n,\tag{2.9}$$

有

$$\begin{aligned}\sum_{t=1}^{q}\sum_{p=1}^{l}\left|s_i(\theta)+s_j(\theta)\right| &= \left(\sum_{t=1}^{t_0-1}+\sum_{t=t_0}^{t_0+1}+\sum_{t=t_0+2}^{q}\right)\left(\sum_{p=1}^{l}\left|s_i(\theta)+s_j(\theta)\right|\right) \\ &= O\left(\sum_{t=1}^{t_0-1}\frac{l}{N^2(\theta_{2lt_0}-\theta_{2lt})^2}+1+\sum_{t=t_0+2}^{q}\frac{l}{N^2(\theta_{2lt_0+l}-\theta_{2l(t-1)})^2}\right) \\ &= O(1).\end{aligned}\tag{2.10}$$

至于$-\pi\le\theta\le 0$, 由于 $s_k(\theta)$是奇函数, 采用以上估计方法, 同样得出(2.10)的结果, 综合以上分析可知(2.4)对于$-\pi\le\theta\le\pi$是成立的

3 定理的证明

由定理2知定理1成立, 故下证定理2 由(1.8)和(1.9), 有

$$W_n(f;r,\theta)-f(\theta)=\sum_{k=1}^{n}f(\theta_k)s_k(\theta)+\sum_{t=1}^{q}B_{2lt}s_{2lt}(\theta)-f(\theta)$$

$$= \{\sum_{k=1}^{n} f(\theta_k)s_k(\theta) + \sum_{k=1}^{n}\frac{1}{2^r}\tilde{\Delta}_h^r f(\theta_k)s_k(\theta) - f(\theta)\} +$$

$$\sum_{t=1}^{q}\frac{1}{2^r}\sum_{p=1}^{l-1}\tilde{\Delta}_h^r f(\theta_{2l(t-1)+2p})(s_{2lt}(\theta) - s_{2l(t-1)+2p}(\theta)) -$$

$$\sum_{t=1}^{q}\frac{1}{2^r}\sum_{p=1}^{l}\tilde{\Delta}_h^r f(\theta_{2l(t-1)+2p-1})(s_{2lt}(\theta) + s_{2l(t-1)+2p-1}(\theta)) -$$

$$\sum_{p=2lq+1}^{n}\frac{1}{2^r}\tilde{\Delta}_h^r f(\theta_p)s_p(\theta) \overset{\triangle}{=} \sum_{j=1}^{4} C_j$$

用 $p(\theta)$ 表示与 $f(\theta)$ 具有最小偏差且次数不高于 n 的奇三角多项式, 便有

$$|p(\theta) - f(\theta)| \le E_n(f), \tag{3.2}$$

这里 $E_n(f)$ 是 $f(\theta)$ 的最佳逼近. 根据 $\tilde{\Delta}_h^r f(\theta)$ 的定义和性质可知 $\tilde{\Delta}_h^r p(\theta)$ 也是次数不高于 n 的奇三角多项式, 从而 $p(\theta)$ 和 $\tilde{\Delta}_h^r p(\theta)$ 与它们自身的内插多项式重合, 即

$$\sum_{k=1}^{n}(\tilde{\Delta}_h^r p(\theta_k) + p(\theta_k))s_k(\theta) = \tilde{\Delta}_h^r p(\theta) + p(\theta). \tag{3.3}$$

先估计 C_1, 由(3.3), 有

$$C_1 = \{\sum_{k=1}^{n}\frac{1}{2^r}\tilde{\Delta}_h^r(f(\theta_k) - p(\theta_k))s_k(\theta) + \sum_{k=1}^{n}(f(\theta_k) - p(\theta_k))s_k(\theta)\} +$$

$$\{\frac{1}{2^r}(\tilde{\Delta}_h^r p(\theta) - \tilde{\Delta}_h^r f(\theta)) + (p(\theta) - f(\theta))\} +$$

$$\frac{1}{2^r}\tilde{\Delta}_h^r f(\theta) \overset{\triangle}{=} \sum_{j=1}^{3} B_j. \tag{3.4}$$

由(1.7), 有

$$B_1 = \frac{1}{2}\{\sum_{k=1}^{n}\frac{1}{2^r}\Delta_h^r(f(\theta_k) - p(\theta_k))s_k(\theta) + \sum_{k=1}^{n}(f(\theta_k) - p(\theta_k))s_k(\theta)\} +$$

$$\frac{1}{2}\{\sum_{k=1}^{n}\frac{1}{2^r}\nabla_h^r(f(\theta_k) - p(\theta_k))s_k(\theta) + \sum_{k=1}^{n}(f(\theta_k) - p(\theta_k))s_k(\theta)\}$$

$$\overset{\triangle}{=} B_{11} + B_{12}, \tag{3.5}$$

$$B_{11} = \frac{1}{2}\{\sum_{k=1}^{r}(f(\theta_k) - p(\theta_k))(\frac{1}{2^r}\sum_{i=0}^{k-1}(-1)^{1+i}\binom{r}{i}s_{k-i}(\theta) + s_k(\theta)) +$$

$$\sum_{k=1+r}^{n}(f(\theta_k) - p(\theta_k))\frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}(-1)^{1+i}s_{k-i}(\theta) + s_k(\theta)) +$$

$$\sum_{k=n+1}^{n+r}(f(\theta_k) - p(\theta_k))(\frac{1}{2^r}\sum_{i=k-n}^{r}\binom{r}{i}(-1)^{1+i}s_{k-i}(\theta))\}. \tag{3.6}$$

由(3.2), (2.1)及(2.9), 得

$$B_{11} = O(E_n(f)) \tag{3.7}$$

$$B_{12} = \frac{1}{2}\{\sum_{k=1-r}^{0}(f(\theta_k) - p(\theta_k))\frac{1}{2^r}\sum_{i=1-k}^{r}(-1)^{1+i}\binom{r}{i}s_{k+i}(\theta) +$$

$$\sum_{k=1}^{n-r}(f(\theta_k)-p(\theta_k))\frac{1}{2^r}\sum_{i=0}^{r}\binom{r}{i}((-1)^{1+i}s_{k+i}(\theta)+s_k(\theta))+$$

$$\sum_{k=n-r+1}^{n}(f(\theta_k)-p(\theta_k))(\frac{1}{2^r}\sum_{i=0}^{n-k}(-1)^{1+i}\binom{r}{i}s_{k+i}(\theta)+s_k(\theta))\}.$$

由(3.2),(2.2)和(2.9),得

$$B_{12}=O(E_n(f)). \tag{3.8}$$

综合B_{11}和B_{12},知

$$B_1=O(E_n(f)). \tag{3.9}$$

利用导数与等距差分之间的关系,有

$$\begin{aligned}&\Delta_h^j f(\theta)=(-1)^{r+1}(\frac{\pi}{N})^j f^{(j)}(\xi_1),\quad \theta<\xi_1<\theta+jh,\\ &\nabla_h^j f(\theta)=-(\frac{\pi}{N})^j f^{(j)}(\xi_2),\quad \theta-jh<\xi_2<\theta\end{aligned} \tag{3.10}$$

于是使用(3.10),有

$$\begin{aligned}\frac{1}{2^{r+1}}\Delta_h^r f(\theta)&=\frac{1}{2^{r+1}}\sum_{i=0}^{r-1-j}(-1)^i\binom{r-1-j}{i}(\Delta_h^j f(\theta+(r-j-i)h)-\Delta_h^j f(\theta+(r-j-i-1)h))\\ &=\frac{(-1)^{1+r}}{2^{1+r}}(\frac{\pi}{N})^j\sum_{i=0}^{r-1-j}(-1)^i\binom{r-1-j}{i}(f^{(j)}(\xi_1)-f^{(j)}(\xi_2))\\ &=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})),\end{aligned} \tag{3.11}$$

其中$0\leq j\leq r-1$, $\theta+(r-j-i)h<\xi_1<\theta+(r-i)h$, $\theta+(r-j-i-1)h<\xi_2<\theta+(r-i-1)h$. 同理可证

$$\frac{1}{2^{r+1}}\nabla_h^r f(\theta)=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})), \tag{3.12}$$

因此

$$B_3=\frac{1}{2^r}\tilde{\Delta}_h^r f(\theta)=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})). \tag{3.13}$$

由(3.2),(1.5),(1.6)和(1.7)很容易得出

$$B_2=O(E_n(f)). \tag{3.14}$$

综合B_1,B_2和B_3,知

$$C_1=O(E_n(f)+\frac{1}{n^j}\omega(f^j,\frac{1}{n})). \tag{3.15}$$

由(3.13),(2.3)和(2.4),有

$$C_2=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})),\quad C_3=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})). \tag{3.16}$$

使用(3.13)和(2.9),有

$$C_4=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})). \tag{3.17}$$

当$f(\theta)\in C_{2\pi}^j$时,由Jackson 定理,有

$$E_n(f)=O(\frac{1}{n^j}\omega(f^j,\frac{1}{n})). \tag{3.18}$$

由(3.18)，综合 C_1, C_2, C_3 和 C_4 便知定理2获证.

定理3的证明 仍设 $p(\theta)$ 为 $f(\theta)$ 的最佳逼近奇三角多项式.

$$W_n(f;r,\theta) - f(\theta) = \{S_n(f-p;\theta) + (p(\theta) - f(\theta)\} + \sum_{l=1}^{q} B_{2lt} S_{2lt}(\theta)$$

$$\overset{\triangle}{=} D_1 + D_2 \tag{3.19}$$

由(3.2)和(3.18)，不难得出

$$D_1 = O\left(\frac{1}{n^r}\omega(f^r, \frac{1}{n})(1+\lambda_n(\theta))\right). \tag{3.20}$$

由 B_{2lt} 的定义和(3.13)，得

$$D_2 = \sum_{l=1}^{q}\sum_{p=1}^{2l}(-1)^p \frac{1}{2^r}\tilde{\Delta}_h^r f(\theta_{2l(t-1)+p}) S_{2lt}(\theta)$$

$$= O\left(\frac{1}{n^r}\omega(f^r, \frac{1}{n})\lambda_n(\theta)\right). \tag{3.21}$$

综合 D_1 和 D_2，可知定理3获证.

注 对于以 $\{\theta_k = \frac{k\pi}{n}\}_{k=0}^{n}$ 为插值节点组所构造的偶三角插值多项式

$$C_n(f;\theta) = \sum_{k=0}^{n} f(\theta_k) c_k(\theta)$$

其中

$$c_k(\theta) = \frac{(\cos\theta - \cos\theta_0)\cdots(\cos\theta - \cos\theta_{k-1})(\cos\theta - \cos\theta_{k+1})\cdots(\cos\theta - \cos\theta_n)}{(\cos\theta_k - \cos\theta_0)\cdots(\cos\theta_k - \cos\theta_{k-1})(\cos\theta_k - \cos\theta_{k+1})\cdots(\cos\theta_k - \cos\theta_n)}^{[1]}$$

$$= \begin{cases} -\dfrac{\sin\theta\sin N\theta}{2n(\cos\theta - 1)}, & k=0 \\ (-1)^{k+1}\dfrac{\sin\theta\sin n\theta}{n(\cos\theta - \cos\theta_k)}, & k = 1, 2, \cdots, n-1 \\ (-1)^{n+1}\dfrac{\sin\theta\sin n\theta}{2n(\cos\theta + 1)}, & k = n \end{cases}$$ [2]

按本文的修正方法对其进行修正也同样能获得定理1至定理3的结论.

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On a Modifying Triangle Interpolation Polynomial

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Abstract

The paper introduces a modifing triangle interpolation polynomial $W_n(f; r, \theta)$ (where r is a given natural number) based on these values of $f(\theta)$ (where $f(\theta) \in C_{2\pi}$ and $f(\theta)$ is an odd function) on these nodes $\{\theta_k = \frac{k}{n+1}\pi\}_{k=1}^{n}$. $W_n(f; r, \theta)$ uniformly converges to $f(\theta)$ ($f(\theta) \in C_{2\pi}$ and $f(\theta)$ is an odd function) on the total real axis. The approximation order of $W_n(f; r, \theta)$ reaches the best approximation order when used to approximate to $f(\theta)$ where $f(\theta) \in C_{2\pi}^{j}$ $(0 \le j \le r-1)$ and $f(\theta)$ is an odd function.

Keywords modifing triangle interpolation polynomial, uniform convergence, best convergence order.

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Iterative Solutions and Its Error Estimation of Nonlinear Equation for Accretive Operator in Banach Spaces

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Abstract

In this paper, we study problems of iterative approximation and its error estimation to nonlinear equation for Lipschitzan strongly accretive operator in Banach Spaces.

Keywords strongly accretive operator, strictly pseudocontractive mapping, Ishikawa iteration process, error estimation.