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Completely Positive Matrices Having Cyclic Graphs *

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Abstract: We prove that a CP matrix A having cyclic graph has exactly two minimal rank 1 factorization if $\det M(A) > 0$ and has exactly one minimal rank 1 factorization if $\det M(A) = 0$.

Key words: completely positive matrix; cyclic graph; minimal rank 1 factorization.

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An $n \times n$ matrix A is said to be completely positive, or in short CP, if there exist k nonnegative column vectors $b_1, b_2, \cdots, b_k \in \mathcal{R}^n$ such that

$$A = b_1 b_1^T + b_2 b_2^T + \cdots + b_k b_k^T. \tag{1}$$

The smallest such number k, denoted by $CP\text{rank}A$, is called the factorization index of A and (1) is called the minimal rank 1 factorization of A if $k = CP\text{rank}A$. For a given $n \times n$ symmetric matrix A, the graph $G(A) = (V, E)$ of A is defined by $V(G(A)) = [n] = \{1, 2, \cdots, n\}$ and $E(G(A)) = \{(i,j) : i, j \in [n] \text{ and } a_{ij} \neq 0\}$. A graph G is completely positive if each of its doubly nonnegative matrix realizations is completely positive. For a real matrix A, the comparison matrix of A, denoted by $M(A)$, is defined to be the matrix whose diagonal entries are the absolute values of those of A and whose off-diagonal entries are the negatives of the absolute values of those of A. Let A be an $n \times n$ matrix and let $\alpha, \beta \subset [n], \alpha, \beta \neq \phi$. We denote by $A[\alpha|\beta]$ the submatrix of A whose rows are indexed by α and whose columns are indexed by β in their natural orders and denote $A[\alpha|\alpha]$ by $A[\alpha]$. In addition, the determinants of $A[\alpha|\beta]$ and $A[\alpha]$ are denoted by $A(\alpha|\beta)$ and $A(\alpha)$, respectively.

In this paper, we prove that a completely positive matrix having cyclic graph has exactly two the minimal rank 1 factorization if $\det M(A) > 0$, and has exactly one the

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minimal rank 1 factorization if $\det M((A) = 0$.

Lemma 1[2] *Let H be a matrix in $Z^{n\times n}$, in other words, all the entries of off-diagonal of H are nonpositive. Then H is an M-matrix if and only if all of principal minors of H are nonnegative.*

Theorem 2 *Let A be an $n\times n$ nonnegative matrix whose graph is a cycle of order $n\geq 4$. Then A is CP if and only if $\det M(A)\geq 0$. Moreover if A is CP, then $CP\mathrm{rank}A = n$.*

Proof Suppose A is CP. Clearly $G(A)$ is triangle-free since $G(A)$ is a cycle of order $n\geq 4$. Hence $M(A)$ is an M-matrix by [3]. Thus $\det M(A)\geq 0$ by Lemma 1.

Conversely suppose $\det M(A)\geq 0$. Then $A_i = A[1,\cdots,i-1,i+1,\cdots,n]$ is a doubly nonnegative matrix whose graph is a tree of order $n-1$ for each $i\in[n]$. So A_i is CP by [1]. Hence $M(A_i)$ is an M-matrix of order $n-1$ by [3]. Furthermore, all principal minors of $M(A_i)$ are nonnegative by Lemma 1. Obviously each principal minor of less than order n in $M(A)$ is a principal minor of some $M(A_i)$. In addition, $det M(A)\geq 0$. Hence all of principal minors of $M(A)$ are nonnegative. Therefore $M(A)$ is an M-matrix by Lemma 1. Thus $A\in CP$ by [3].

Finally, we prove that $CP\mathrm{rank}A = n$. Since A is CP and $G(A)$ is triangle-free, A has a minimal rank 1 factorization (1) in which each b_i has at most 2 positive entries by [3]. Hence A has at most $2k$ positive off-diagoral entries. In addition, A has $2n$ positive off-diagonal entries because $G(A)$ is a cycle of order n. Hence $2k\geq 2n$, i.e. $CP\mathrm{rank}A = k\geq n$. On the other hand, since $G(A)$ is triangle-free and $M(A)$ is an M-matrix, $CP\mathrm{rank}A\leq \max\{|V(G(A))|,|E(G(A))|\} = n$ by [3]. and $|V(G(A))| = |E(G(A))| = n$. This shows that $CP\mathrm{rank}A = n$.

Remark 3 It is easy to see from Theorem 2 and its proof that, in a minimal rank 1 factorization of A, each b_i has exactly two positive entries and the subscripts of the b_i's can be chosen so that the inner product of b_i and b_{i+1} is positive for $i\in[n]$, where $b_{n+1}\equiv b_1$.

Let A be a doubly nonnegative matrix whose graph is a cycle of order $n\geq 4$. Without loss of generality, we may suppose

$$A = \begin{pmatrix} a_{11} & a_{12} & 0 & \cdots & 0 & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & 0 & 0 \\ 0 & a_{32} & a_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n1} & 0 & 0 & \cdots & a_{n,n-1} & a_{nn} \end{pmatrix}. \tag{2}$$

Lemma 4 *Let A be a doubly nonnegative matrix in the form (2). Then*

(i) $a_{11}M(A)(2,\cdots,n) - a_{12}^2 M(A)(3,\cdots,n) = \det M(A) + a_{1n}^2 M(A)(2,\cdots,n-1) + 2a_{12}a_{23}\cdots a_{n-1,n}a_{n1}$,

(ii) $M(A)(1,2,\cdots,n-1)M(A)(2,\cdots,n) = M(A)(2,\cdots,n-1)\det M(A) + (a_{12}a_{23}\cdots a_{n-1,n} + a_{1n}M(A)(2,\cdots,n-1))^2$,

(iii) $\dfrac{M(A)(1,\cdots,n-2)}{M(A)(1,\cdots,n-1)}\geq\dfrac{M(A)(2,\cdots,n-2)}{M(A)(2,\cdots,n-1)}\geq\cdots\geq\dfrac{M(A)(n-2)}{M(A)(n-2,n-1)}\geq\dfrac{1}{a_{n-1,n-1}}$.

Proof Clearly,

$$\det M(A) = a_{11}M(A)(2,\cdots,n) + a_{12}\begin{vmatrix} -a_{21} & -a_{23} & \cdots & 0 & 0 \\ 0 & a_{33} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -a_{n1} & 0 & \cdots & -a_{n,n-1} & a_{nn} \end{vmatrix} +$$

$$(-1)^n a_{1n}\begin{vmatrix} -a_{21} & a_{22} & -a_{23} & \cdots & 0 \\ 0 & -a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{n-1,n-1} \\ -a_{n1} & 0 & 0 & \cdots & -a_{n,n-1} \end{vmatrix}$$

$$= a_{11}M(A)(2,\cdots,n) - a_{12}^2 M(A)(3,\cdots,n) - a_{12}a_{23}\cdots a_{n-1,n}a_{n1} - a_{1n}^2 M(A)(2,\cdots,n-1) - a_{12}a_{23}\cdots a_{n-1,n}a_{n1}.$$

Thus (i) holds.

We define a 2×2 matrix $S=(s_{ij})$ as follows:

$$s_{11} = M(A)(1,\cdots,n-1), \quad s_{12} = M(A)(1,\cdots,n-1|2,\cdots,n),$$
$$s_{21} = M(A)(2,\cdots,n|1,\cdots,n-1), \quad s_{22} = M(A)(2,\cdots,n).$$

It follows from sylvester's identity (e.g.[4]) that

$$\det S = M(A)(2,\cdots,n-1)\det M(A).$$

On the other hand

$$\det S = s_{11}s_{22} - s_{12}s_{21} = M(A)(1,\cdots,n-1)M(A)(2,\cdots,n) - (a_{12}a_{23}\cdots a_{n-1,n} + a_{1n}M(A)(2,\cdots,n-1))^2.$$

Thus (ii) holds.

By using the similar method in (ii), it is easy to prove that $M(A)(1,\cdots,n-2)M(A)(2,\cdots,n-1) \geq M(A)(1,\cdots,n-1)M(A)(2,\cdots,n-2)$. i.e. the first inequality of (iii) holds. The others are proved similarly.

Let A be a CP matrix in form (2). Then $CP\mathrm{rank}A=n$ by Theorem 2. We may assume $b_1=(b_{11},b_{21},0,\cdots,0)^T, b_2=(0,b_{22},b_{32},0,\cdots,0)^T,\cdots, b_n=(b_{1n},0,\cdots,0,b_{nn})^T$ and $b_{ii}, b_{i,i+1}$ are positive for each $i\in[n]$ by Remark 3.

Theorem 5 *Let A be a CP matrix having cyclic graph. Then A has exactly two minimal rank 1 factorizations if* $\det M(A)>0$, *and has exactly one minimal rank 1 factorization if* $\det M(A)=0$.

Proof Without loss of generality, let A be in form (2). It is easy to see that the number of minimal rank 1 factorizations of A is equal to the number of positive solutions of the following equations:

$$\begin{array}{ll} b_{11}^2 + b_{1n}^2 = a_{11}, & b_{11}b_{21} = a_{12}, \\ b_{21}^2 + b_{22}^2 = a_{22}, & b_{22}b_{32} = a_{23}, \\ \cdots & \cdots \\ b_{n,n-1}^2 + b_{nn}^2 = a_{nn}, & b_{nn}b_{1n} = a_{1n}. \end{array} \qquad (3)$$

Put $b_{nn}=x$. Then it follows from (3) that

$$b_{n,n-1}=\sqrt{a_{nn}-x^2},\qquad b_{n-1,n-1}=\frac{a_{n-1n}}{\sqrt{a_{nn}-x^2}},$$

$$b_{k,k-1}=\sqrt{\frac{M(A)(k,\cdots,n)-M(A)(k,\cdots,n-1)x^2}{M(A)(k+1,\cdots,n)-M(A)(k+1,\cdots,n-1)x^2}},\quad b_{k-1,k-1}=\frac{a_{k-1,k}}{b_{k,k-1}},$$

$$\begin{aligned}b_{1n} &= \sqrt{\frac{a_{11}M(A)(2,\cdots,n)-a_{12}^2M(A)(3,\cdots,n)-M(A)(1,\cdots,n-1)x^2}{M(A)(2,\cdots,n)-M(A)(2,\cdots,n-1)x^2}}\\ &= \sqrt{\frac{\det M(A)+a_{1n}^2M(A)(2,\cdots,n-1)+2a_{12}\cdots a_{n1}-M(A)(1,\cdots,n-1)x^2}{M(A)(2,\cdots,n)-M(A)(2,\cdots,n-1)x^2}},\end{aligned}$$

$k=n-1,\cdots,2,(M(A)(\phi)\equiv 1)$, by using Lemma 4(i). In addition $b_{1n}=\frac{a_{1n}}{x}$. Therefore, we have the following equation

$$\begin{aligned}&M(A)(1,\cdots,n-1)x^4-[\det M(A)+2a_{1n}^2M(A)(2,\cdots,n-1)+\\ &2a_{12}a_{23}\cdots a_{n-1,n}a_{n1}]x^2+a_{1n}^2M(A)(2,\cdots,n)=0.\end{aligned}\tag{4}$$

The discriminant of equation (4) is

$$\begin{aligned}\Delta &= [\det M(A)+2a_{1n}^2M(A)(2,\cdots,n-1)+2a_{12}\cdots a_{n1}]^2-\\ &\quad 4a_{1n}^2M(A)(1,\cdots,n-1)M(A)(2,\cdots,n)\\ &= (\det M(A))^2+4a_{12}\cdots a_{n1}\det M(A)\geq 0\end{aligned}$$

by using Lemma 4(ii) and Theorem 2. Since $a_{1n}^2M(A)(2,\cdots,n)>0$, it is easy to see that (4) has exactly two positive solutions if $\det M(A)>0$, and has exactly one positive solution if $\det M(A)=0$. Furthermore for each positive solution x of (4), we have

$$\begin{aligned}&M(A)(k,\cdots,n)-M(A)(k,\cdots,n-1)x^2\\ &=a_{nn}M(A)(k,\cdots,n-1)-a_{n-1,n}^2M(A)(k,\cdots,n-2)-M(A)(k,\cdots,n-1)\times\\ &\quad\frac{\det M(A)+2a_{1n}^2M(A)(2,\cdots,n-1)+2a_{12}\cdots a_{n1}\pm\sqrt{\Delta}}{2M(A)(1,\cdots,n-1)}\\ &=\frac{M(A)(k,\cdots,n-1)}{2M(A)(1,\cdots,n-1)}[2a_{nn}M(A)(1,\cdots,n-1)-(\det M(A)+\\ &\quad 2a_{1n}^2M(A)(2,\cdots,n-1)+2a_{12}\cdots a_{n1}\pm\sqrt{\Delta})]-a_{n-1,n}^2M(A)(k,\cdots,n-2)\\ &=\frac{M(A)(k,\cdots,n-1)}{2M(A)(1,\cdots,n-1)}\times\\ &\quad[(2\det M(A)+2a_{1n}^2M(A)(2,\cdots,n-1)+2a_{n-1,n}^2M(A)(1,\cdots,n-2)+4a_{12}\cdots a_{n1})-\\ &\quad(\det M(A)+2a_{1n}^2M(A)(2,\cdots,n-1)+2a_{12}\cdots a_{n1}\pm\sqrt{\Delta})]-a_{n-1,n}^2M(A)(k,\cdots,n-\\ &=\frac{M(A)(k,\cdots,n-1)}{2M(A)(1,\cdots,n-1)}(\det M(A)+2a_{12}\cdots a_{n1}\mp\sqrt{\Delta})+\\ &\quad a_{n-1,n}^2[\frac{M(A)(1,\cdots,n-2)M(A)(k,\cdots,n-1)}{M(A)(1,\cdots,n-1)}-M(A)(k,\cdots,n-2)]\\ &>0,\qquad k=n-1,\cdots,2\end{aligned}$$

by Lemma 2.4 (i) and (iii). Therefore the number of positive solutions of (4) is equal to the number of positive solutions of (3). Thus the result holds.

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图是圈的完全正矩阵

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摘 要: 本文证明，图是圈的完全正矩阵 A 当比较矩阵 $M(A)$ 的行列式大于零时，恰有两个极小秩 1 分解，而当 $\det M(A)=0$ 时，恰有一个极小秩 1 分解.