Journal of Mathematical Research & Exposition
Vol.20, No.1, 37-42, February, 2000

BMO and Singular Integrals over Certain Disconnected Groups *

ZHU Yue-ping[1], *ZHENG Wei-xing*[2]

(1. Dept. of Math., Nantong Teachers' College, 226007;
2. Dept. of Math., Nanjing University, 210093)

Abstract: In this paper the authors study proprieties of certain convolution operators on $L^\infty(G)$ and weighted $\mathrm{BMO}(\alpha)(G)$ spaces,where G is a locally compact totally disconnected group with a suitable sequence of open compact subgroups. The authors prove that if the kernel satisfies certain conditions, then the convolution operator is bounded from L^∞ to $\mathrm{BMO}(\alpha)$ or from $\mathrm{BMO}(\alpha)$ to $\mathrm{BMO}(\alpha)$.

Key words: singular integral; BMO; weighted inequality.

Classification: AMS(1991) 42B25/CLC O174.2

Document code: A **Article ID**: 1000-341X(2000)01-0037-06

1. Definitions and notation

Throughout this paper, G will denote a locally compact Abelian topological group with a suitable collection of open compact subgroups in the sense of Edwards and Gaudry[2]. This means that there exists a strictly decreasing sequence $\{G_n\}_{n\in Z}$ of open compact subgroups of G such that

(i) $G_{n+1}\in G_n$ and $\sup\{\mathrm{order}(G_n/G_{n+1})\}<\infty$;

(ii) $\cup_{-\infty}^{\infty}G_n=G$ and $\cap_{-\infty}^{\infty}G_n=\{0\}$;

(iii) $\mu(G_0)=1,\mu$ denotes the Haar measure on G.

Such groups are the locally compact analogue of the so-called Vilenkin groups which were first described by N.Ya Vilenkin in 1947. Examples of such groups are the additive group of the p-adic numbers and,more generally, of a local field, see [1].

Let Γ denote the dual group of G, and Γ_n the annihilator of G_n in Γ for each $n\in \mathbf{Z}$. Then $(\Gamma_n)_{-\infty}^{\infty}$ is a strictly increasing sequence of open compact subgroup of Γ such that

(iv) $\mathrm{order}(\Gamma_{n+1}/\Gamma_n)=\mathrm{order}(G_n/G_{n+1})$;

(v) $\cup_{-\infty}^{\infty}\Gamma_n=\Gamma$ and $\cap_{-\infty}^{\infty}\Gamma_n=\{1\}$;

(vi) $\lambda(\Gamma_0)=1$, where λ denotes the Haar measure on Γ.

---

*Received date: 1997-03-26
Biography: ZHU Yue-ping (1965-), female, associate professor.

Let $\mu(G_n) = (\lambda(\Gamma_n))^{-1} = m_n^{-1}$, for each $n \in \mathbf{Z}$. For all $\alpha > 0, k \in \mathbf{Z}$ we have ([3])

$$\sum_{n=k}^{\infty} m_n^{-\alpha} \leq C m_k^{-\alpha}, \tag{1}$$

$$\sum_{n=-\infty}^{k} m_n^{\alpha} \leq C m_k^{\alpha}. \tag{2}$$

For $\alpha \in \mathbf{R}$, we define the function $v_\alpha : G \to \mathbf{R}$ by

$$v_\alpha(x) = \begin{cases} m_n^{-\alpha}, & \text{for } x \in G_n \setminus G_{n+1}, \\ 0, & \text{for } x = 0. \end{cases}$$

If G is the additive group of a local field, $v_1(x) = |x|$ and, hence $v_\alpha(x) = |x|^\alpha$ for all $x \in G$. We denote the L_p spaces with respect to the measures $\mu_\alpha = v_\alpha \mathrm{d}\mu$ on G by $L_{p,\alpha}(G)$. $L_{p,\alpha}(G) = \{f : f$ is a measurable function on G and

$$\|f\|_{p,\alpha} = (\int_G |f(x)^p v_\alpha(x) \mathrm{d}\mu(x))^{\frac{1}{p}} < \infty\} (1 \leq p < \infty).$$

We say a locally integrable function f has bounded mean oscillation,i.e., $f \in \mathrm{BMO}(\alpha)$, if

$$f_{\mu_\alpha}^{\#}(x) = \sup_n \frac{1}{\mu_\alpha(x+G_n)} \int_{x+G_n} |f(y) - f_{x+G_n}| v_\alpha(x) \mathrm{d}\mu(x) < \infty, \text{a. e. on } G, \tag{3}$$

where for a measurable set B of positive measure, $f_B = \frac{1}{\mu_\alpha(B)} \int_B f(x) v_\alpha(x) \mathrm{d}\mu(x)$. We let $\|f\|_{\mathrm{BMO}(\alpha)} = \|f_{\mu_\alpha}^{\#}\|_\infty$. An equivalent norm for $\mathrm{BMO}(\alpha)$ is obtained if the L_1-norm in (3) is replaced by the L_p-norm for any $p \in (1, \infty)$.

Let $k \in L_{\mathrm{loc}}(G \setminus \{0\})$ such that the integral operator T defined by

$$Tf(x) = \text{p. v.} \int k(x-y) f(y) \mathrm{d}\mu(y) \tag{4}$$

is bounded on $L_{2,\alpha}$. We say k satisfies condition $C_r, 1 < r < \infty$, if k is locally in L_r on $G \setminus \{0\}, k$ has mean value zero, and there exist $C, \varepsilon > 0$ such that for all $l, n \in \mathbf{Z}$ with $n < l$ we have

$$\sup_{y \in G_l} (\int_{G_n \setminus G_{n+1}} |k(x-y) - k(x)|^r \mathrm{d}\mu(x))^{\frac{1}{r}} \leq C m_n^{\varepsilon + \frac{1}{r'}} m_l^{-\varepsilon} \text{ if } 1 < r < \infty \tag{5}$$

and there exists $C > 0$ so that for all $l \in \mathbf{Z}$ we have

$$\sup_{y \in G_l} \int_{G \setminus G_l} |k(x-y) - k(x)| \mathrm{d}\mu(x) \leq C \text{ if } r = 1. \tag{6}$$

For example, if $G = (K, +), K$ is a local field with the notation of ([1]), $G_n = \mathcal{P}^n$, a large class of kernels k that satisfy the condition C_1 are smooth homogeneous kernels $k(x) = \frac{\Omega(x)}{|x|}$ where $\Omega(\beta^j x) = \Omega(x)$ for $x \neq 0, j \in \mathbf{Z}, \int_{|x|=1} \Omega(x) \mathrm{d}\mu(x) = 0$, and $\sup_{|y|=1} \sum_{j=1}^{\infty} \int_{|x|=1} |\Omega(x + \beta^j y) - \Omega(x)| \mathrm{d}\mu(x) < \infty$.

In this paper, we consider more general classes of kernels defined by C_r and give direct proofs of the boundedness of the corresponding operator on BMO(α).

2. Estimates on BMO(α) for Convolution Operator

Throughout this part we denote $x+G_n$ by G'_n, and the weighted mean of f on G_n by f_{G_n}.

Theorem 2.1 *Let $f \in L^\infty$ and be supported on a set of finite measure. If $k \in C_r(1 \le r < \infty)$, then Tf exists a.e., $Tf \in$ BMO(α) and $\|Tf\|_{\mathrm{BMO}(\alpha)} \le C\|f\|_\infty$, where C is independent of f.*

Proof Since $f \in L_\infty \subset L_{2,\alpha}$, Tf exists a.e. . Let $E = \{x \in G : Tf(x) \text{ exists }\}$ and x_0 be a point of density of E. For $n_0 \in Z$, consider the set $G'_{n_o} = x_0 + G_{n_0}$. Write f as $f(x) = f_{G'_{n_0}} + [f(x) - f_{G'_{n_0}}]\chi_{G'_{n_0}}(x) + [f(x) - f_{G'_{n_0}}]\chi_{(G'_{n_0})^c}(x) = f_1 + f_2(x) + f_3(x)$. Since f_1 is a constant, $T(f_1) = 0$ and hence exists a.e.. Using the fact T is bounded on $L_{2,\alpha}$, we know

$$\int_{G'_{n_0}} |Tf_2(x)|\nu_\alpha(x)\mathrm{d}\mu(x) \le \mu_\alpha(G'_{n_0})^{\frac{1}{2}}\|Tf_2\|_{2,\alpha} \le C\mu_\alpha(G'_{n_0})^{\frac{1}{2}}\|f_2\|_{2,\alpha} \le C\mu_\alpha(G'_{n_0})\|f\|_\infty.$$

Thus Tf_2 exists a.e.. There is a point $y_0 \in G'_{n_0}$ such that $Tf_3(y_0)$ exists,

$$Tf_3(y_0) = Tf(y_0) - Tf_2(y_0).$$

For $x \in G'_{n_0}$,

$$\begin{aligned}
&|Tf_3(x) - Tf_3(y_0)| \\
&\le \int |k(x-z) - k(y_0 - z)||f_3(z)|\mathrm{d}\mu(z) \\
&= \int_{G\setminus G'_{n_0}} |k(x-z) - k(y_0-z)||f(z) - f_1|\mathrm{d}\mu(z) \\
&\le \Big(\int_{G\setminus G'_{n_0}} |k(x-z) - k(y_0-z)|^r \mathrm{d}\mu(z)\Big)^{\frac{1}{r}}\Big(\int_{G\setminus G'_{n_0}} |f(z) - f_1|^{r'}\mathrm{d}\mu(z)\Big)^{\frac{1}{r'}} \\
&= \sum_{n=-\infty}^{n_0-1}\Big(\int_{G'_n\setminus G'_{n+1}} |k(x-z) - k(y_0-z)|^r \mathrm{d}\mu(z)\Big)^{\frac{1}{r}}\Big(\int_{G'_n\setminus G'_{n+1}} |f(z) - f_1|^{r'}\mathrm{d}\mu(z)\Big)^{\frac{1}{r'}} \\
&\le \sum_{n=-\infty}^{n_0-1} m_n^{\epsilon+\frac{1}{r'}} m_{n_0}^{-\epsilon}\|f\|_\infty \mu(G'_{n-1}\setminus G'_n)^{\frac{1}{r'}} \\
&\le C\|f\|_\infty \sum_{n=-\infty}^{n_0} m_n^\epsilon m_{n_0}^{-\epsilon} \le C\|f\|_\infty.
\end{aligned}$$

So $Tf \in$ BMO(α), and $\|Tf\|_{\mathrm{BMO}(\alpha)} \le C\|f\|_\infty$. □

Theorem 2.1 can be extended to the case $f \in$ BMO(α), if we require additional smoothness of the kernel k. We say k satisfies the C_r^+ condition if we replace condition (5) with

$$\sup_{y\in G_l}\Big(\int_{G_n\setminus G_{n+1}} |k(x-y) - k(x)|^r \mathrm{d}\mu(x)\Big)^{\frac{1}{r}} \le C(l-n)^{-1} m_n^{\epsilon+\frac{1}{r'}} m_l^{-\epsilon} (1 < r < \infty). \qquad (7)$$

We have

Theorem 2.2 *If k satisfies C_r^+ for some r, $1 \le r < \infty$, and $f \in \mathrm{BMO}(\alpha), -1 < \alpha < 0$. Then either Tf exists only on a set of measure zero or $Tf \in \mathrm{BMO}(\alpha)$ with $\|Tf\|_{\mathrm{BMO}(\alpha)} \le C\|f\|_{\mathrm{BMO}(\alpha)}$ where C is independent of f.*

Before we show Theorem 2.2, we first introduce two basic lemmas.

Lemma 2.3[3] *Let $\alpha > -1, x \in G$ and $k \in \mathbf{Z}$.*

(a) $\mu_\alpha(G_k) \sim m_k^{-(1+\alpha)}$;

(b) $\mu_\alpha(x + G_k) \le C\mu_\alpha(x + G_{k+1})$;

(c) $\mu_\alpha(G_k) \le C\mu_\alpha(G_k \setminus G_{k+1})$;

(d) *If $\alpha \le 0, (G_k')^* = G_k' \setminus \{0\}$, then $\mu_\alpha(G_k') \le Cm_k^{-1}\inf\{v_\alpha(y) : y \in (G_k')^*\}$.*

Lemma 2.4 *Let $1 \le p < \infty$. There is a constant C such that if $f \in \mathrm{BMO}(\alpha), x_0 \in G, n \ge 1$, then*

$$\int_{G'_{j-n}} |f(y) - f_{G'_j}|^p \mathrm{d}\mu_\alpha(y) \le C^p n^p \mu_\alpha(G'_{j-n})\|f\|^p_{\mathrm{BMO}(\alpha)}, \forall j \in Z,$$

where $G'_j = G_j + x_0$.

Proof First consider the case $n = 1$. We have

$$\begin{aligned}&\{\int_{G'_{j-1}} |f(y) - f_{G'_j}|^p \mathrm{d}\mu_\alpha(y)\}^{\frac{1}{p}}\\ &\le \{\int_{G'_{j-1}} |f(y) - f_{G'_{j-1}}|^p \mathrm{d}\mu_\alpha(y)\}^{\frac{1}{p}} + \mu_\alpha(G'_{j-1})^{\frac{1}{p}}|f_{G'_{j-1}} - f_{G'_j}|\\ &\le C\mu_\alpha(G'_{j-1})^{\frac{1}{p}}\|f\|_{\mathrm{BMO}(\alpha)} + \mu_\alpha(G'_{j-1})^{\frac{1}{p}}|f_{G'_{j-1}} - f_{G'_j}| \le C\mu_\alpha(G'_{j-1})^{\frac{1}{p}}\|f\|_{\mathrm{BMO}(\alpha)}.\end{aligned}$$

We now proceed by induction. Suppose

$$\int_{G'_{j-n}} |f(y) - f_{G'_j}|^p \mathrm{d}\mu_\alpha(y) \le Cn^p \mu_\alpha(G'_{j-n})\|f\|^p_{\mathrm{BMO}(\alpha)}.$$

For G'_{j-n-1},

$$\begin{aligned}&(\int_{G'_{j-n-1}} |f(y) - f_{G'_j}|^p \mathrm{d}\mu_\alpha(y))^{\frac{1}{p}}\\ &\le (\int_{G'_{j-n-1}} |f(y) - f_{G'_{j-n}}|^p \mathrm{d}\mu_\alpha(y))^{\frac{1}{p}} + \mu_\alpha(G'_{j-n-1})^{\frac{1}{p}}|f_{G'_{j-n}} - f_{G'_j}|\\ &\le C\mu_\alpha(G'_{j-n-1})^{\frac{1}{p}}\|f\|_{\mathrm{BMO}(\alpha)} + Cn\|f\|_{\mathrm{BMO}(\alpha)}\mu_\alpha(G'_{j-n-1})^{\frac{1}{p}}\\ &= C(n+1)\mu_\alpha(G'_{j-n-1})^{\frac{1}{p}}\|f\|_{\mathrm{BMO}(\alpha)}. \quad \square\end{aligned}$$

Proof of Theorem 2.2 Let $E = \{x \in G : Tf(x) \text{ exists }\}$. Assume E has positive measure. Using the same notation as in the proof of Theorem 2.1, we write $f(x) = f_1 + f_2(x) + f_3(x)$. Similar to the proof of Theorem 2.1,we have

$$\int_{G'_{n_0}} |Tf_2(x)|\mathrm{d}\mu_\alpha(x) \le \mu_\alpha(G'_{n_0})^{\frac{1}{2}}\|Tf_2\|_{2,\alpha} \le C\mu_\alpha(G'_{n_0})^{\frac{1}{2}}\|f_2\|_{2,\alpha}$$
$$= C\mu_\alpha(G'_{n_0})^{\frac{1}{2}}\{\int_{G'_{n_0}} |f(x) - f_{G_{n_0}}|^2\mathrm{d}\mu_\alpha(x)\}^{\frac{1}{2}}$$
$$\le C\mu_\alpha(G'_{n_0})\|f\|_{\mathrm{BMO}(\alpha)}.$$

For the estimate $|Tf_3(x) - Tf_3(y_0)|$, we use the fact $k \in C_r^+$. Then we have

$$|Tf_3(x) - Tf_3(y_0)|$$
$$\le \sum_{n=-\infty}^{n_0-1} \{\int_{G'_n\setminus G'_{n+1}} |k(x-z) - k(y_0 - z)|^r \mathrm{d}\mu(z)\}^{\frac{1}{r}}\{\int_{G'_n\setminus G'_{n+1}} |f(z) - f_1|^{r'}\mathrm{d}\mu(z)\}^{\frac{1}{r'}}$$
$$\le C\sum_{n=-\infty}^{n_0-1} (n_0 - n)^{-1} m_n^{\varepsilon+\frac{1}{r'}} m_{n_0}^{-\varepsilon}\{\int_{G'_n} |f(z) - f_{G'_{n_0}}|^{r'}\mathrm{d}\mu(z)\}^{\frac{1}{r'}}$$
$$\le C\sum_{n=-\infty}^{n_0-1} (n_0 - n)^{-1} m_n^{\varepsilon+\frac{1}{r'}} m_{n_0}^{-\varepsilon}\{\int_{G'_n} |f(z) - f_{G'_{n_0}}|^{r'} v_\alpha(z)\mathrm{d}\mu(z) m_n^{-1}(\mu_\alpha(G'_n))^{-1}\}^{\frac{1}{r'}}$$
$$\le \sum_{n=-\infty}^{n_0-1} (n_0 - n)^{-1} m_n^{\varepsilon} m_{n_0}^{-\varepsilon}(n_0 - n)\|f\|_{\mathrm{BMO}(\alpha)}$$
$$\le C\|f\|_{\mathrm{BMO}(\alpha)}.$$

So

$$\frac{1}{\mu_\alpha(G'_{n_0})}\int_{G'_{n_0}} |Tf(x) - Tf_3(y_0)|\mathrm{d}\mu_\alpha(x) \le C\|f\|_{\mathrm{BMO}(\alpha)},$$

i.e.,

$$\|Tf\|_{\mathrm{BMO}(\alpha)} \le C\|f\|_{\mathrm{BMO}(\alpha)}. \quad \square$$

Corollary 2.5 *Let ϕ be a bounded radial function on Γ. For each integer n, let $\phi_n = \phi\chi_{\Gamma_n\setminus\Gamma_{-n}}$, and let F_n be the function on G such that $\hat{F}_n = \phi_n$. Then*

$$\|F_n * f\|_{\mathrm{BMO}(\alpha)} \le C\|f\|_{\mathrm{BMO}(\alpha)},$$

where C is independent of f.

References:

[1] TAIBLESON M. T. *Fourier Analysis on Local Fields* [M]. Math. Notes, Princeton University Press, Princeton, 1975.

[2] EDWARDS R E and GANDRY G I. *Littlewood-Paley Operator and Multiplier Theory* [M]. Springer-Verlag, Berlin, 1977.

[3] ONNEWEER C W. *Multiliers on weighted L_p spaces over certain totally disconnected groups* [J]. Trans. Amer. Math. Soc., 1985, **288**: 347-362.

[4] DALY J E and KURTZ D S. *BMO and singular integrals over local fields* [J]. J. Austral. Math. Soc. (Series A), 1993, **54**: 321-333.

[5] KURTZ D S. *Littlewood-Paley operators on BMO* [J]. Proc. Amer. Math. Soc., 1987, **99**: 657-666.

一类全不连通群上的 BMO 和奇异积分算子

朱 月 萍[1]， 郑 维 行[2]

(1. 南通师范学院数学系， 226007;

2. 南京大学数学系， 210093)

摘 要: 本文主要讨论了定义在局部紧的全不连通群 G 上的一类卷积算子在加权 $L^\infty(G)$ 和 $BMO(\alpha)$ 空间的性态. 证明了如果卷积算子的核满足适当的条件，则算子是 $L^\infty(G)$ 到 $BMO(\alpha)$ 有界的或是 $BMO(\alpha)$ 到 $BMO(\alpha)$ 有界的.