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The Singularly Perturbed Nonlinear Boundary Value Problem *

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Abstract: The singularly perturbed nonlinear problem

$$\epsilon^2 y'' - f(x, y, y') = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,$$

$$y(0) = A,$$

$$ay'(1) + y(1) = B,$$

where y, f, A, B are n-dimensional vectors is considered. Under the appropriate assumptions the authors prove that there exists a solution $y(x,\epsilon)$ and the estimation of $y(x,\epsilon)$ is obtained using the method of differential inequalities.

Key words: singular perturbation; differential inequality; boundary value problem.

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Consider the singularly perturbed nonlinear problem for $y = (y_1, \cdots, y_n)$ of the form

$$\varepsilon^2 y'' - f(x, y, y') = 0, \quad 0 < x < 1,\ 0 < \varepsilon \ll 1, \tag{1}$$

$$y(0) = A, \tag{2}$$

$$ay'(1) + by(1) = B, \tag{3}$$

where ε is a small positive parameter, $f = (f_1, \cdots, f_n)$, $A = (A_1, \cdots, A_n)$, $B = (B_1, \cdots, B_n)$ and $a \geq 0, b \geq 0, ab > 0$. Many authors such as Howes[1], O'Donnell[2], Chen[4],Mo[5–8] and others have discussed this problem under various situations using the method of differential inequalities. In this paper the authors consider the case of $f_{iy'_j} \geq 0$ and $f_{iy'_j} \leq 0$.

Theorem *Assume the following:*

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[a] The reduced problem of (1)-(3)

$$f(x,y,y')=0,\quad 0<x<1,$$

$$y(0)=A$$

has a solution $y_0(x)=(y_{10}(x),\cdots,y_{n0}(x))\in C^2[0,1]$ *and* $y'_{i0}>0, B_i\geq ay'_{i0}(1)+by_{i0}(1)$.

[b] $f_i(x,y,y')\equiv f_i(x,y_i,\cdots,y_n,z_1,\cdots,z_n)\in C^2([0,1]\times\Pi_{i=1}^n[a_i,b_i]\times R^n$ *and there are positive nondecreasing continuous* $\phi_i, i=1,2,\cdots,n$ *on* $[0,\infty)$ *such that*

$$|\ f_i(x,y,z)\ |\leq\phi_i(|\ z_i\ |),\quad i=1,2,\cdots,n$$

and

$$\int^{\infty}(s/\phi_i(s))ds=\infty,\quad i=1,2,\cdots,n.$$

[c] There are positive constants $l_i,\ i=1,2,\cdots,n$ *such that* $f_{iy_iz_i}(x,y,z)\geq l_i$ *and* $f_{iy_iz_j}(x,y,z)\geq 0,\ i\neq j,k,\ f_{iz_j}(x,y,z)$ *and* $f_{iy_j}(x,y,0)$ *are nonnegative functions.*

Then for ε *small enough the problem (1)-(3) has a solution* $y(x,\varepsilon)\equiv(y_1(x,\varepsilon),\cdots,y_n(x,\varepsilon))$ $\in C^2[0,1]$ *which satisfies*

$$y_i(x,\varepsilon)=y_{i0}(x)+O\left(\frac{\varepsilon^2(\sqrt{b^2\varepsilon^2+2al_iC_i}-b\varepsilon)}{2a\varepsilon+(\sqrt{b^2\varepsilon^2+2al_iC_i}-b\varepsilon)(1-x)}\right)+O(\varepsilon),$$

$$0<x<1,\ 0<\varepsilon\ll 1,\tag{4}$$

where $C_i=B_i-(ay'_{i0}(1)+by_{i0}(1))$.

Proof We first construct functions $\alpha_i(x,\varepsilon),\ \beta_i(x,\varepsilon),\ i=1,2,\cdots,n$ on $x\in[0,1]$:

$$\alpha_i(x,\varepsilon)=y_{i0}-r_i\varepsilon,\tag{5}$$

$$\beta_i(x,\varepsilon)=y_{i0}+w_i(x,\varepsilon)+r_i\varepsilon,\tag{6}$$

where $r_i,\ i=1,2,\cdots,n$ are large enough positive constants which will be determined below, $w_i(x,\varepsilon),\ i=1,2,\cdots,n$ are positive functions possessing boundary layer behavior near $x=1$ and they satisfy

$$\varepsilon^2w_i''-l_iw_iw_i'=0,\quad 0<x<1,$$

$$aw_i'(1,\varepsilon)+bw_i(1,\varepsilon)=C_i.$$

In fact, we can select $w_i(x,\varepsilon)$ that

$$w_i(x,\varepsilon)=\frac{2\varepsilon^2(\sqrt{b^2\varepsilon^2+2al_iC_i}-b\varepsilon)}{l_i[2a\varepsilon+(\sqrt{b^2\varepsilon^2+2al_iC_i}-b\varepsilon)(1-x)]}.\tag{7}$$

Obviously, from (7) we have

$$w_i(x,\varepsilon)\geq 0,\quad w_i'(x,\varepsilon)\geq 0,\quad 0\leq x\leq 1.\tag{8}$$

For arbitrary positive ε and r_i we have

$$\alpha_i(x,\varepsilon) \in C^2[0,1], \quad \beta(x,\varepsilon) \in C^2[0,1], \tag{9}$$

$$\alpha_i(x,\varepsilon) < \beta_i(x,\varepsilon), \quad 0 \leq x \leq 1, \tag{10}$$

and

$$\alpha_i(0,\varepsilon) = A_i - r_i\varepsilon,$$

$$\beta_i(0,\varepsilon) = y_{i0}(0) + w_i(0,\varepsilon) + r_i\varepsilon \geq A_i + r_i\varepsilon,$$

$$a\alpha_i'(1,\varepsilon) + b\alpha_i(1,\varepsilon) = ay_{i0}'(1) + by_{i0}(1) - br_i\varepsilon \leq B_i - br_i\varepsilon,$$

$$a\beta_i'(1,\varepsilon) + b\beta_i(1,\varepsilon) = ay_{i0}'(1) + by_{i0}(1) + aw_i'(1,\varepsilon) + bw_i(1,\varepsilon) + br_i\varepsilon = B_i + br_i\varepsilon.$$

Thus

$$\alpha_i(0,\varepsilon) \leq A_i \leq \beta_i(0,\varepsilon), \tag{11}$$

$$a\alpha_i'(1,\varepsilon) + b\alpha_i(1,\varepsilon) \leq B_i \leq a\beta_i'(1,\varepsilon) + b\beta_i(1,\varepsilon), \tag{12}$$

From the mean value theorem and the assumptions of the theorem there are $\xi_j,\ \eta_i \in (y_{jo}, y_{j0} + w_j + r_j\varepsilon)$ and $\xi_j',\ \eta_j' \in (y_{j0}', y_{j0}' + W_j'),\ j = 1,2,\cdots,n$ such that

$$\begin{aligned} f_i(x,\beta,\beta') \ &\geq \ \sum_{j,k=1}^{n} f_{iy_jz_k}(x,\xi_i,\cdots,\xi_n,\xi_1',\cdots,\xi_n') \times (w_j + r_j\varepsilon)(y_{k0}' + w_k') + \\ &\quad \sum_{k=1}^{n} f_{iz_k}(x,\eta_1,\cdots,\eta_n,\eta_1',\cdots,\eta_n')w_k' \\ &\geq \ l_i(w_i + r_i\varepsilon)(y_{i0}' + w_i'). \end{aligned}$$

From $y_{i0} \in C^2[0,1],\ y_{i0}' > 0$ on $x \in [0,1]$, there are positive constants m_{1i} and m_{2i} such that

$$|y_{i0}"| \leq m_{1i}, \quad y_{i0}' \geq m_{2i}, \quad i = 1,2,\cdots,n.$$

Hence

$$\varepsilon^2\beta_i" - f_i(x,\beta,\beta') \leq \varepsilon^2 y_{i0}" + \varepsilon^2 w_i" - l_i(w_i + r_i\varepsilon)(y_{i0}' + w_i') \leq (m_{1i} - l_i m_{2i} r_i)\varepsilon.$$

Furthermore, as $r_i \geq r_{i0} = m_{1i}/(l_i m_{2i}),\ i = 1,2,\cdots,n$, we obtain

$$\varepsilon^2\beta_i" - f_i(x,\beta,\beta') \leq 0, \quad x \in (0,1), \quad i = 1,2,\cdots,n. \tag{13}$$

Analogously, we have

$$f_i(x,\alpha,\alpha') \geq -l_i r_i \varepsilon y_{i0}', \quad i = 1,2,\cdots,n,$$

and hence

$$\varepsilon^2\alpha_i" - f_i(x,\alpha,\alpha') \geq \varepsilon^2 y_{i0}" + l_i r_i \varepsilon y_{i0}', \geq (-m_{1i} + l_i m_{2i} r_i)\varepsilon.$$

Therefore, as $r_i \geq r_{i0} = m_{1i}/(l_i m_{2i})$, $i = 1, 2, \cdots, n$, we have

$$\varepsilon^2 \alpha_i'' - f_i(x, \alpha, \alpha') \geq 0 \quad x \in (0,1), \quad i = 1, 2, \cdots, n. \tag{14}$$

Thus we have (9)-(14) as $r_i \geq r_{i0}$. Now from theory of differential inequalities [3], the boundary value problem (1)-(3) has a solution $y(x,\varepsilon) \equiv (y_1(x,\varepsilon), \cdots, y_n(x,\varepsilon)) \in C^2$and there holds

$$\alpha_i(x,\varepsilon) \leq y_i(x,\varepsilon) \leq \beta_i(x,\varepsilon), \quad i = 1, 2, \cdots, n, \ 0 \leq x \leq 1. \tag{15}$$

Substituting (5)-(6) into (15), we obtain (4). The theorem is proved.

Example Consider the singularly perturbed boundary value problem

$$\varepsilon^2 y'' - yy' + 1 = 0, \quad 0 < x < 1, \ 0 < \varepsilon \ll 1, \tag{16}$$

$$y(0) = 1, \tag{17}$$

$$y'(1) + y(1) = 4, \tag{18}$$

where $f(x, y, y') \equiv yy' - 1 \in C^2([0,1] \times [1,4] \times R)$.

The reduced problem of (16)-(18)

$$yy' - 1 = 0,$$

$$y(0) = 1$$

has a solution $y_0(x) = (2x+1)^{1/2} \in C^2[0,1]$.

Obviously, $y_0(x)$ and $f(x,y,y')$ satisfy the assumptions $(a) - (c)$ of the theorem. So there exists a solution $y(x,\varepsilon) \in C^2[0,1]$ of the problem (16)-(18), which satisfies

$$y(x,\varepsilon) = (2x+1)^{1/2} + O\left(\frac{\varepsilon^2(\sqrt{\varepsilon^2 + 2C} - \varepsilon)}{2\varepsilon + (\sqrt{\varepsilon^2 + 2C} - \varepsilon)(1-x)}\right) + O(\varepsilon),$$

$$0 < x < 1, \ 0 < \varepsilon \ll 1,$$

where $C = 4(1 - 3^{-1/2})$.

Remark In the assumptions of the above theorem if $f_{iz_j} \leq 0$ and $y'_{i0} < 0$, then we could also obtain corresponding results.

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奇摄动非线性边值问题

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摘 要: 本文研究了奇摄动非线性边值问题

$$\varepsilon^2 y" - f(x,y,y') = 0,\ 0 < x < 1,\ 0 < \epsilon \ll 1,$$

$$y(0) = A,$$

$$ay'(1) + y(1) = B,$$

其中 y, f, A, B 为 n- 维向量. 在适当的假设下作者利用微分不等式方法证明了存在一个解 $y(x,\epsilon)$, 并得到了它的估计式.