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Ishikawa Iterative Process with Errors for Lipschitzian and ϕ-Hemicontractive Mappings in Banach Spaces *

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Abstract: Let X be a real Banach space, K a nonempty convex subset of X such that $K+K\subset K$. Let $T:K\to K$ be a Lipschitzian and ϕ-hemicontractive mapping with a Lipschitzian constant $L\geq 1$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $[0,1]$ satisfying: (i) $\alpha_n\to 0$, $\beta_n\to 0$ as $n\to\infty$; (ii) $\sum\limits_{n=0}^{\infty}\alpha_n=\infty$. Assume that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in K satisfying $\|u_n\|=o(\alpha_n)$, $v_n\to 0$ as $n\to\infty$. For an arbitrary $x_0\in K$ define a sequence $\{x_n\}_{n=0}^{\infty}$ in K by

$$\text{(IS)}\quad\begin{cases} x_{n+1}=(1-\alpha_n)x_n+\alpha_nTy_n+u_n, \\ y_n=(1-\beta_n)x_n+\beta_nTx_n+v_n,\ n\geq 0. \end{cases}$$

If $\{Ty_n\}$ is bounded, then the sequence $\{x_n\}$ converges strongly to the unique fixed point of T.
A related result deals with iterative solution of nonlinear equations with ϕ-strongly quasi-accretive mappings by the Ishikawa iteration with errors in an arbitrary Banach space.

Key words: Ishikawa iteration with errors; ϕ-strongly quasi-accretive mapping; ϕ-hemicontraction, arbitrary Banach space.

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1. Introduction and Preliminaries

Let X be a real Banach space with norm $\|\cdot\|$ and a dual X^*. The normalized duality mapping $J:X\to 2^{X^*}$ is defined by

$$Jx=\{x^*\in X^*:<x,x^*>=\|x\|^2=\|x^*\|^2\},$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. It is well known that if X^* is strictly convex, then J is single-valued and such that $J(tx)=tJx$ for all $t\geq 0$, $x\in X$. If X^*

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is uniformly convex, then J is uniformly continuous on any bounded subsets of X (cf. Browder[1], Barbu[3]).

An operator T with domain $D(T)$ and range $R(T)$ in X is said to be accretive if for every $x, y \in D(T)$, there exists a $j \in J(x-y)$ such that

$$\langle Tx - Ty, j \rangle \geq 0. \tag{1}$$

The concept of accretive operators was introduced independently by Browder[1] and Kato[2] in 1967. A fundamental and important result, due to Browder, in the theory of accretive operators states that the IVP

$$\frac{\mathrm{d}u}{\mathrm{d}t} + Tu = 0,\ u(0) = u_0 \tag{2}$$

is solvable if T is a locally Lipschitzian and accretive operator on X. An accretive operator T is strongly accretive if there exists a positive constant k such that the inequality (1) holds with 0 replaced by $k\|x-y\|^2$. Without loss of generality, we may assume that $k \in (0,1)$. These operators have been studied by various authors (cf. [4,5,6]). Deimling[4] proved that if X is a Banach space, and $T : X \to X$ is continuous and strongly accretive, then $R(T) = X$. Hence, for any $f \in X$, the equation $Tx = f$ has at least one solution in X. Since T is strongly accretive, the solution must be unique.

A class of mappings that are more general than strongly accretive ones is the class of ϕ–strongly quasi–accretive ones. A mapping T is said to be ϕ–strongly quasi–accretive, if the kernel of T, $N(T) = \{x \in D(T) : Tx = 0\} \neq \emptyset$, and there exists an increaing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geq \phi(\|x-y\|)\|x-y\|. \tag{3}$$

A class of mappings closely related to ϕ–strongly quasi–accretive mappings is so called the kind of ϕ–hemicontractions. A mapping T is called ϕ–hemicontractive, if $(I-T)$ is ϕ–strongly quasi–accretive, where $I : X \to X$ denotes the identity mapping. Such mappings have been used and studied by several authors (e.g., cf. Xu and Roach[13], Zhou and Jia[10], Osilike[12]).

Recently, Osilike proved that both the Mann iteration method and the Ishikawa iteration methods are applied to approximate the fixed points of ϕ–hemicontractive mappings in a real q–uniformly smooth Banach space.

Theorem A *Let $q > 1$, and let E be a real q–uniformly smooth Banach space. Let $T : X \to X$ be a Lipschitzian ϕ–strongly accretive operator. Suppose that the equation $Tx = f$ has a solution for any given $f \in X$. Let $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ be real sequences satisfying*

(i) $0 < \alpha_n < 1, n \geq 0$;
(ii) $0 \leq \beta_n \leq \alpha_n^{q-1}, n \geq 0;,$
(iii) $\sum_{n=0}^{\infty} \alpha_n(1-\alpha_n)^{q-1} = \infty$;
(iv) $\sum_{n=0}^{\infty} \alpha_n^q < \infty$.
Define $S : X \to X$ by $Sx = f + x - Tx$, for each $x \in X$.

Then the sequence $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in X$ *by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n S x_n, n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n S y_n, n \geq 0 \end{aligned}$$

converges strongly to the unique solution of the equation $Tx = f$.

Theorem B *Let* $q > 1$, *and let* E *be a real* q*–uniformly smooth Banach space. Let* K *be a nonempty closed convex subset of* E *and* $T : K \to K$ *be a Lipschitzian* ϕ*–hemicontractive operator. Let* $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ *be real sequences satisfying*

(i) $0 < \alpha_n < 1, n \geq 0$;
(ii) $0 \leq \beta_n \leq \alpha_n^{q-1}, n \geq 0;$,
(iii) $\sum_{n=0}^{\infty} \alpha_n (1-\alpha_n)^{q-1} = \infty$;
(iv) $\sum_{n=0}^{\infty} \alpha_n^q < \infty$. *Then the sequence* $\{x_n\}_{n=0}^{\infty}$ *generated from any* $x_0 \in K$ *by*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, n \geq 0 \end{aligned}$$

converges strongly to the fixed point of T.

Indeed, Theorem A of Osilike[12] can be deduced from the above Theorem B. To see this, assume that all conditions are satisfied in the Theorem 1 of Osilike[12], let $Sx = f + (I-T)x$, then $S : E \to E$ is Lipschitzian ϕ-hemicontractive. By Theorem B we obtain the desired conclusion.

On the other hand, Theorem 13 of Chidume[7] proved that the Ishikawa iteration process converges strongly to the unique fixed point of T when E is any real smooth Banach space and T is a Lipschitzian strongly pseudocontractive mapping from a nonempty closed convex subset K of E to itself.

One question arises naturally: Is it possiible to extend Theorems A,B of Osilike[12] to the case where X is a real Banach spaces without any smoothness?

In this paper we shall solve this question in the more general setting. To do so, we need the following known result.

Lemma 1.1 *Let* X *be a real Banach space. Then for each* $x, y \in X$, $j(x+y) \in J(x+y)$,

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle.$$

Proof It follows from the fact that $Jx = \partial\phi(x)$, where $\phi(x) = \frac{1}{2}\|x\|^2$. □

2. Main Results

Now we prove the main results of this paper.

Theorem 2.1 *Let* X *be a real Banach space,* K *a nonempty convex subset of* X *such that* $K + K \subset K$. *Let* $T : K \to K$ *be a Lipschitzian and* ϕ*–hemicontractive mapping with a Lipschitzian constant* $L \geq 1$. *Let* $\{\alpha_n\}_{n=0}^{\infty}$ *and* $\{\beta_n\}_{n=0}^{\infty}$ *be two real sequences in* $[0,1]$ *satisfying:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ *as* $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Assume that $\{u_n\}_{n=0}^{\infty}$ *and* $\{v_n\}_{n=0}^{\infty}$ *are two sequences in* K *satisfying* $\|u_n\| = o(\alpha_n)$, $v_n \to 0$ *as* $n \to \infty$.

For an arbitrary $x_0 \in K$ *define a sequence* $\{x_n\}_{n=0}^{\infty}$ *in* K *by*

$$(\text{IS})_1 \begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n \\ y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, \, n \geq 0. \end{cases}$$

If $\{Ty_n\}$ *is bounded, then the sequence* $\{x_n\}$ *converges strongly to the unique fixed point of* T.

Proof Since $K + K \subset K$, and K is convex, we see that the sequence $\{x_n\}$ is well-defined.

By the defination of T, we know that T has a unique fixed point in K. Let q denote the unique fixed point and $L \geq 1$ denote the Lipschitzian constant of T.

Now we shall show that $\{x_n\}$ is bounded. In fact, since $\|u_n\| = o(\alpha_n)$, we have $\|u_n\| = \epsilon_n \alpha_n$, where $\epsilon_n \to 0$ as $n \to \infty$. Let $d = \sup_{n \geq 0}\{\|Ty_n - q\| + \epsilon_n\} + \|x_0 - q\|$. Then, by a simple induction, we can show $\|x_n - q\| \leq d$, for all $n \geq 0$.

Since T is ϕ–hemicontractive, we have

$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|, \tag{4}$$

for each $x, y \in K$.

By using Lemma 1.1 and $(\text{IS})_1$ we get

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \|(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq) + u_n\|^2 \\ &\leq \|(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 + 2\langle u_n, j(x_{n+1} - q)\rangle \\ &\leq \|(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 + 2d\|u_n\|. \end{aligned} \tag{5}$$

Again using Lemma 1.1 and $(\text{IS})_1$, we obtain that

$$\begin{aligned} &\|(1-\alpha_n)(x_n - q) + \alpha_n(Ty_n - Tq)\|^2 \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle Ty_n - Tq, j(x_{n+1} - q - u_n)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\langle Ty_n - T(x_{n+1} - u_n), j(x_{n+1} - u_n - q)\rangle + \\ &\quad 2\alpha_n\langle T(x_{n+1} - u_n) - Tq, j(x_{n+1} - u_n - q)\rangle \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n L\|y_n - x_{n+1} - u_n\|\|x_{n+1} - u_n - q\| + \\ &\quad 2\alpha_n\|x_{n+1} - u_n - q\|^2 - 2\alpha_n\phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\| \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n L\{[\alpha_n(1+L^2) + \beta_n(1+L)]\|x_n - q\| + (\alpha_n L + 1)\|v_n\|\} \times \\ &\quad (L^2\|x_n - q\| + \alpha_n L\|v_n\|) + 2\alpha_n\|x_{n+1} - u_n - q\|^2 - \\ &\quad 2\alpha_n\phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\| \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n L\{[\alpha_n(1+L^2) + \beta_n(1+L)]d + (\alpha_n L + 1)\|v_n\|\} \times \\ &\quad (L^2 d + \alpha_n L\|v_n\|) + 2\alpha_n\|x_{n+1} - u_n - q\|^2 - \\ &\quad 2\alpha_n\phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\| \\ &\leq (1-\alpha_n)^2\|x_n - q\|^2 + 2\alpha_n\tau_n + 2\alpha_n\|x_{n+1} - u_n - q\|^2 - \\ &\quad 2\alpha_n\phi(\|x_{n+1} - u_n - q\|)\|x_{n+1} - u_n - q\|, \end{aligned} \tag{6}$$

where $\tau_n = L\{[\alpha_n(1+L^2)+\beta_n L]d + (\alpha_n L + 1)||v_n||\}(L^2 d + \alpha_n L||v_n||)$.

It follows from (6) that

$$\begin{aligned}||x_{n+1}-u_n-q||^2 &\le \frac{(1-\alpha_n)^2}{1-2\alpha_n}||x_n-q||^2 + \frac{2\alpha_n\tau_n}{1-2\alpha_n} - \\ &\quad \frac{2\alpha_n}{1-2\alpha_n}\phi(||x_{n+1}-u_n-q||)||x_{n+1}-u_n-q|| \\ &\le ||x_n-q||^2 + \frac{2\alpha_n}{1-2\alpha_n}(\frac{d^2\alpha_n}{2}+\tau_n) - \\ &\quad \frac{2\alpha_n}{1-2\alpha_n}\phi(||x_{n+1}-u_n-q||)||x_{n+1}-u_n-q||. \end{aligned} \tag{7}$$

Substituting (7) into (5) yields

$$\begin{aligned}||x_{n+1}-q||^2 &\le \frac{(1-\alpha_n)^2}{1-2\alpha_n}||x_n-q||^2 + \frac{2\alpha_n\tau_n}{1-2\alpha_n} - \\ &\quad \frac{2\alpha_n}{1-2\alpha_n}\phi(||x_{n+1}-u_n-q||)||x_{n+1}-u_n-q|| + 2d||u_n|| \\ &\le ||x_n-q||^2 + \frac{2\alpha_n}{1-2\alpha_n}(\frac{d^2\alpha_n}{2}+\tau_n) - \\ &\quad \frac{2\alpha_n}{1-2\alpha_n}\phi(||x_{n+1}-u_n-q||)||x_{n+1}-u_n-q|| + 2d||u_n||. \end{aligned} \tag{8}$$

Now we consider two possible cases.

Case (1). $\inf_{n\ge 0}\{||x_{n+1}-u_n-q||\} = \delta > 0$.

Since $d^2\alpha_n + 2\tau_n + 2d\epsilon_n(1-2\alpha_n) \to 0$ as $n\to\infty$, we see that there exists some fixed N such that

$$d^2\alpha_n + 2\tau_n + (2d\epsilon_n)(1-2\alpha_n) < \phi(\delta)\delta, \tag{9}$$

for all $n \ge N$.

It follows from (8) and (9) that

$$\begin{aligned}||x_{n+1}-q||^2 &\le ||x_n-q||^2 + \frac{\alpha_n}{1-2\alpha_n}\phi(\delta)\delta - \frac{2\alpha_n}{1-2\alpha_n}\phi(\delta)\delta \\ &\le ||x_n-q||^2 - \frac{\alpha_n}{1-2\alpha_n}\phi(\delta)\delta. \end{aligned} \tag{10}$$

(10) leads to

$$\phi(\delta)\delta\sum_{n=N}^{\infty}\alpha_n < ||x_N-q||^2 < \infty, \tag{11}$$

which contradicts the assumption that $\sum_{n=0}^{\infty}\alpha_n = \infty$. This contradiction shows the case (1) is impossble.

Case (2). $\inf_{n\ge 0}\{||x_{n+1}-u_n-q||\} = 0$.

In this case, there exists a subsequence $\{x_{n_j+1}\}$ such that $x_{n_j+1} \to q$ as $j \to \infty$. Hence, $\forall \epsilon > 0$, there exists some fixed $n_j \geq 0$ such that

$$\|x_{n_j+1} - q\| < \epsilon,\ d^2\alpha_n + 2\tau_n + 2d\epsilon_n(1 - 2\alpha_n) < \phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2},\ \|u_n\| < \frac{\varepsilon}{2},$$

for all $n \geq n_j$.

Now we want show that $\|x_{n_j+m} - q\| < \varepsilon$, for all $m \geq 1$.

We first show that $\|x_{n_j+2} - q\| < \varepsilon$. If not, assume that $\|x_{n_j+2} - q\| \geq \varepsilon$, then

$$\|x_{n_j+2} - u_{n_j+1} - q\| \geq \|x_{n_j+2} - q\| - \|u_{n_j+1}\| \geq \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2},$$

and hence $\phi(\|x_{n_j+2} - u_{n_j+1} - q\|) \geq \phi(\frac{\varepsilon}{2})$.

By (9) we have

$$\|x_{n_j+2} - q\|^2 \leq \|x_{n_j+1} - q\|^2 - \frac{\alpha_{n_j+1}}{1 - 2\alpha_{n_j+1}}\phi(\frac{\varepsilon}{2})\frac{\varepsilon}{2} < \|x_{n_j+1} - q\|^2,$$

a contradition. This contradiction shows $\|x_{n_j+2} - q\| < \epsilon$. By using induction, we can show $\|x_{n_j+m} - q\| < \epsilon$, for all $m \geq 1$, which gives to $x_n \to q$ as $n \to \infty$. The proof of Theorem 2.1 is complete. □

Remark 1 Theorem 2.1 extends Theorem 2 of Osilike[12] to the more general Banach spaces without making any smoothness assumption and to the more general iteration with errors. By setting $u_n \equiv 0$, $v_n \equiv 0$, we can deduce Theorem 2 of Osilike[12], and Theorems 4-6,13 of Chidume[7].

Remark 2 Theorem 2.1 also holds true when T is a uniformly continuous and ϕ–hemicontractive mapping.

As a corollary of Theorem 2.1, we have following

Theorem 2.2 *Let X be a real Banach space. Let $T : X \to X$ be a Lipschitzian and ϕ–strongly quasi–accretive mapping with a Lipschitzian constant $L \geq 1$. Set $L_1 = L + 1$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $[0, 1]$ satisfying:*

(i) $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Set $Sx = x - Tx$ for each $x \in X$.

Assume that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two sequences in X satisfying $\|u_n\| = o(\alpha_n)$, and $v_n \to 0$ as $n \to \infty$.

For an arbitrary $x_0 \in X$, an iteration sequence $\{x_n\}$ is defined by

$$(\mathrm{IS})_2 \begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n S y_n + u_n, \\ y_n = (1 - \beta_n)x_n + \beta_n S x_n + v_n,\ n \geq 0. \end{cases}$$

Suppose, furthermore, that $\{Sy_n\}$ is bounded, then the sequence $\{x_n\}$ converges strongly to the unique solution of the equation $Tx = 0$.

Remark 3 Theorem 2.2 extends Theorem 1 of Osilike[12] to the more general Banach spaces and the more general iteration with errors. By setting $u_n \equiv 0$, $v_n \equiv 0$, we can deduce Theorem 1 of Osilike[12].

Remark 4 Theorem 2.2 still holds true when T is a uniformly continuous ϕ–strongly quasi–accretive mapping.

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Banach 空间中关于 Lipschitz ϕ- 半压缩映象的带误差项的 Ishikawa 迭代过程

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摘 要: 本文在任意 Banach 空间中研究了 Lipschitz ϕ- 半压缩映象与 ϕ- 强拟增生映象的带误差项的 Ishikawa 迭代过程，使用新的分析技巧建立了几个强收敛定理.