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Several Properties of Idempoent and Nilpotent Matrices *

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Abstract: Using a limit process, it is proved in this paper that the adjoint matrix of an idempotent matrix is idempotent and the adjoint matrix of a nilpotent matrix is also nilpotent. The results are richer than that in [1].

Key words: idempotent matrix; nilpotent matrix.

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It is known that for an $n \times n$ complex matrix ,denote by $A \in M_n(\mathrm{C})$,if $A^m = A$, $m \geq 2$ is a positive integer, then A is said to be idempotent.Moreover, if $A^m = 0$, $m \geq 1$ then A is said to be nilpotent. In [1],the authors proved that for an $n \times n$ real matrix A, if $A^2 = A$,then $(\mathrm{adj}A)^2 = \mathrm{adj}A$, where,$adjA$ denotes the adjoint matrix of A, but the proof is very sophisticated. In this paper, we prove that this result holds for general idempotent matrix,furthermore, a similar conclusion for nilpotent matrix is also obtained. The proof in this paper is simple and elementary.

Definition 1[2] *Let $A = [a_{ij}] \in M_n(C)$, A^T denotes the transpose of A, $\det A$ denotes the determinant of A and A_{ij} is the algebraic cofactor of in $\det A$, then the matrix*

$$\mathrm{adj}A = [A_{ij}]^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

is called the adjoint matrix of A.

Theorem 1 *Let $A \in M_n(C)$,if $A^m = A$ for a positive integer $m \geq 2$, then* $(\mathrm{adj}A)^m = \mathrm{adj}A$.

Proof Concerning the Jordan form of A, one can easily see that if $A^m = A$, $m \geq 2$, then A is diagonalizable, that is, there exists a nonsingular matrix $T \in M_n(C)$ such that

$$A = T\mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]T^{-1}$$

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where,$\lambda_i(1 \le i \le n)$ satisfy the equation $\lambda^m = \lambda$.

(i) $\mathrm{rank}A = n$. In this case, $\lambda_i \neq 0, i = 1, \cdots, n$. By the equality,

$$A \cdot \mathrm{adj}A = \mathrm{adj}A \cdot A = \det A \cdot I_n.$$

We have $\mathrm{adj}A = \det A \cdot A^{-1} = \lambda_1\lambda_2\cdots\lambda_n T\mathrm{diag}[\lambda_1^{-1}, \cdots, \lambda_n^{-1}]T^{-1}$. Observe that $\lambda_i(1 \le i \le n)$ satisfy the equation $\lambda^{m-1} = 1$, we can get $(\mathrm{adj}A)^{m-1} = I_n$, so that $(adjA)^m = \mathrm{adj}A$.

(ii) $\mathrm{rank}A \le n-2$. All cofactors of order $n-1$ in $\det A$ are zeros, that is $A_{ij} = 0, 1 \le i, j \le n$, i.e.$\mathrm{adj}A = 0$,which implies $(\mathrm{adj}A)^m = \mathrm{adj}A$.

(iii) $\mathrm{rank}A = n-1$. Without loss of generality,we can assume $\lambda_i \neq 0, (1 \le i \le n-1)$ $\lambda_n = 0$. Let $\varepsilon \neq 0$ and $A_\varepsilon = A + \varepsilon T\mathrm{diag}[0, 0, \cdots, 0, 1]T^{-1} = T\mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_{n-1}, \varepsilon]T^{-1}$, then we have

$$\begin{aligned}\mathrm{adj}A_\varepsilon &= \det A_\varepsilon A_\varepsilon^{-1} \\ &= \lambda_1\lambda_2\cdots\lambda_{n-1}\varepsilon T\mathrm{diag}[\lambda_1^{-1}, \cdots, \lambda_{n-1}^{-1}, \varepsilon^{-1}]T^{-1} \\ &= T\mathrm{diag}[\lambda_2\cdots\lambda_{n-1}\varepsilon, \cdots, \lambda_1\cdots\lambda_{n-2}\varepsilon, \cdots, \lambda_1\cdots\lambda_{n-1}]T^{-1},\end{aligned}$$

so that

$$\mathrm{adj}A = \lim_{\varepsilon\to 0}\mathrm{adj}A_\varepsilon = T\mathrm{diag}[0, \cdots, o, \lambda_1\cdots\lambda_{n-1}]T^{-1}.$$

Since $\lambda_i(1 \le i \le n-1)$ are zeros of the equation $\lambda^m = \lambda, (\mathrm{adj}A)^m = \mathrm{adj}A$ follows. This completes the proof.

Corollary 1 *If $A \in M_n(C)$ is idempotent and $A = T\mathrm{diag}[\lambda_1, \lambda_2, \cdots, \lambda_n]T^{-1}$, then*

$$\mathrm{adj}A = T\mathrm{diag}[\prod_{j\neq 1}\lambda_j, \prod_{j\neq 2}\lambda_j, \cdots, \prod_{j\neq n}\lambda_j]T^{-1}.$$

Especially,if $A^2 = A$ and $\mathrm{rank}A = n-1$, we have $A + \mathrm{adj}A = I_n$.

Proof An immediate consequence of the proof of theorem 1 and the fact that $A^2 = A$ implies the eigenvalues of A are either 1 or 0.

The following theorem is devoted to the nilpotent matrices.

Theorem 2 *Let $A \in M_n(C)$. If $A^m = 0$ for a positive integer m, then* $(\mathrm{adj}A)^2 = 0$.

Proof According to the Jordan form of A, all eigenvalues of A are zeros and $\mathrm{rank}A \le n-1$. Similarly to the proof of Theorem 1(ii), we can prove that $\mathrm{adj}A = 0$ and $(\mathrm{adj}A)^2 = 0$ if $\mathrm{rank}A \le n-2$. so it suffices to give a proof for $\mathrm{rank}A = n-1$. In this case, A has the following form:

$$A = T\begin{bmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & 0 \end{bmatrix}T^{-1}$$

let

$$A_\varepsilon = A + \varepsilon I_n = T \begin{bmatrix} \varepsilon & 1 & & & \\ & \varepsilon & \ddots & & \\ & & \ddots & \ddots & \\ & & & \ddots & 1 \\ & & & & \varepsilon \end{bmatrix} T^{-1}$$

$\varepsilon \neq 0$. A direct computation leads to

$$A_\varepsilon^{-1} = T \begin{bmatrix} \frac{1}{\varepsilon} & -\frac{1}{\varepsilon^2} & \cdots & \cdots & (-1)^{n-1}\frac{1}{\varepsilon^n} \\ & \frac{1}{\varepsilon} & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & -\frac{1}{\varepsilon^2} \\ & & & & \frac{1}{\varepsilon} \end{bmatrix} T^{-1}$$

Consequently,

$$\mathrm{adj}A_\varepsilon = \det A_\varepsilon \cdot \ A_\varepsilon^{-1} = \varepsilon^n \cdot A_\varepsilon^{-1}$$

$$\mathrm{adj}A = \lim_{\varepsilon \to 0} \mathrm{adj}A_\varepsilon = T \begin{bmatrix} 0 & 0 & & & (-1)^{n-1} \\ & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & 0 \\ & & & & 0 \end{bmatrix} T^{-1}.$$

In $T^{-1} \cdot \mathrm{adj}A \cdot T$, only $(1, n)$ element is $(-1)^{n-1}$, and the other elements are all zeros, so $(\mathrm{adj}A)^2 = 0$. This proves the theorem.

References:

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幂等阵和幂零阵的伴随阵的若干性质

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摘 要: 本文利用极限过程的方法，证明了幂等阵和幂零阵的伴随矩阵分别是幂等阵和幂零阵. 所得到的结论比 [1] 丰富得多.