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Quadric Completion of Operator Partial Matrices (I) *

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Abstract: For a given quadric polynomial $p(t)$, the necessary and sufficient conditions are obtained for operator partial matrices of the form $\binom{A\ C}{?\ ?}$ to be completed to an operator T such that $p(T) = 0$. Moreover, all such possible completions, if exist, are presented parametrically.

Key words: operator matrix; completion.

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1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. Let $\mathcal{B}(\mathcal{H})$ (resp., $\mathcal{B}(\mathcal{K}, \mathcal{H})$) be the Banach space of all bounded linear operators acting on $\mathcal{H}$ (resp., from $\mathcal{K}$ into $\mathcal{H}$). Motivated by the commutant lifting theory, interpolation theory and control theory, there is recently a growing interest in the study of the completion problem of operator partial matrices. Many deep results have been obtained on the completion of operator partial matrices to projections, contractions, positive operators and so on (see, e.g., [1], [5], [7], [8] and the references therein). The following open problem was raised by P. Rosenthal [1].

Problem R When does $\begin{pmatrix} A & C \\ ? & ? \end{pmatrix}$ (resp., $\begin{pmatrix} A & C \\ ? & B \end{pmatrix}$) have an algebraic completion ? And in particular, is the nilpotent completion problem solvable for $\begin{pmatrix} A & C \\ ? & ? \end{pmatrix}$ (resp., $\begin{pmatrix} A & C \\ ? & B \end{pmatrix}$)?

Recall that an operator T is algebraic if there is a polynomial $p(t)$ such that $p(T) = 0$, and T is nilpotent if there is a positive integer k such that $T^k = 0$. For a given quadric polynomial $p(t)$, we say that an operator partial matrix has a quadric-$p(t)$-completion if it has a completion T such that $p(T) = 0$. We say that an operator partial matrix has a quadric completion if it has a quadric-$p(t)$-completion for some quadric polynomial $p(t)$. Obviously, to answer the Problem R, it is natural to start with the quadric completion problem, that is, the problem that when an operator partial matrix has a quadric completion.

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For any given quadric polynomial $p(t)$, the purpose of this paper is to find the necessary and sufficient conditions for $\binom{A\ C}{?\ ?}$ to have a quadric-$p(t)$- completion, and, if exist, give a characterization of all such completions. The necessary and sufficient conditions are also found for $\binom{A\ C}{?\ ?}$ to have a quadric-$p(t)$-completion with norm not greater than $u \geq r_p = \max\{|\alpha| : p(\alpha) = 0\}$. The similar question for case $\binom{A\ C}{?\ B}$ will be discussed in [2].

Now, we introduce some notations. For a linear manifold $\mathcal{M} \subset \mathcal{K}$, its closure and orthogonal complement will be denoted by $\overline{\mathcal{M}}$ and $\mathcal{M}^{\perp}$ respectively, and $P_{\mathcal{M}}$ will denote the projection onto $\overline{\mathcal{M}}$ along $\mathcal{M}^{\perp}$. For an operator T, denote by ker$T$, R(T), and Lat$T$ the null space, the range, and the lattice of all invariant subspaces of T, respectively. We use $T|_{\mathcal{M}}$to denote the restriction of T to $\mathcal{M}$ which is a linear mapping from $\mathcal{M}$ into $\mathcal{H}$, and T^{-1} for the closed operator defined by

$$T^{-1}x = \begin{cases} (T|_{(\ker T)^{\perp}})^{-1}x & \text{if } x \in R(T); \\ 0 & \text{if } x \in R(T)^{\perp}. \end{cases}$$

Recall that $R(S) \subseteq R(T)$ if and only if $SS^* \leq \lambda^2 TT^*$ for some positive number λ, which happens if and only if $E = T^{-1}S$ is bounded with $\| E \| \leq \lambda$ (ref.[4]).

2. Quadric completion

Let $p(t) = t^2 + at + b$ be a quadric polynomial. Let $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$ be given. In this section we consider the quadric-$p(t)$-completion of $\binom{A\ C}{?\ ?}$.

Lemma 2.1 *Let $p(t) = (t-\alpha)^2$ be a quadric polynomial and $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be given such that $p(T) = 0$ and $p(S) = 0$.*

(i) If at least one of $\mathcal{H}$ and $\mathcal{K}$ is finite dimensional, then the set of all solutions to equation

$$SX + XT = 2\alpha X \tag{1}$$

is the set $\{X = G - SW + WT : (S - \alpha I)G = 0 \text{ and } G(T - \alpha I) = 0, W \text{ arbitrary}\}$.

(ii) If both $\mathcal{H}$ and $\mathcal{K}$ are infinite dimensional, then X is a solution of equation (1) if and only if $X = s - \lim_{n\to\infty}(G_n - SW_n + W_nT)$ for some operator sequences $\{W_n\}$ and $\{G_n\}$ with $(S - \alpha I)G_n = 0$ and $G_n(T - \alpha I) = 0$.

Note: s-lim denotes the limit in the strong operator topology.

Proof Equation (1) is the same as the equation $(S - \alpha I)X + X(T - \alpha I) = 0$. Since both $S - \alpha I$ and $T - \alpha I$ are square-zero, by use of lemma 2.4 in [1]. we know the assertions are true.

Lemma 2.2 *Let $p(t) = (t-\alpha)(t-\beta)$ be a quadric polynomial with $\alpha \neq \beta$. Let $T \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be given such that $p(T) = 0$ and $p(S) = 0$. Then the set of all solutions to equation*

$$SX + XT = (\alpha + \beta)X \tag{2}$$

is exactly the set $\{WT - SW : W \in \mathcal{B}(\mathcal{H},\mathcal{K})\}$.

Proof Let $S_0 = (\beta-\alpha)^{-1}(S-\alpha)$ and $T_0 = (\alpha-\beta)^{-1}(T-\beta)$. Then $S_0^2 = S_0$ and $T_0^2 = T_0$.

Equation (2) becomes $S_0X - XT_0 = 0$. So, by lemma 2.1 of [6], X is a solution to (2) if and only if there exists an operator W such that

$$\begin{aligned} X &= WT_0 - (I - S_0)W = (\alpha-\beta)^{-1}W(T-\beta) - (I - (\beta-\alpha)^{-1}(S-\alpha))W \\ &= (\alpha-\beta)^{-1}[W(T-\beta) - (S-\beta)W] = (\alpha-\beta)^{-1}(WT - SW). \end{aligned}$$

This also means that X is a solution to (2) if and only if $X = WT - SW$ for some W.

Theorem 2.3 *Let $p(t) = t^2 + at + b$ be a quadric polynomial. Let $A \in \mathcal{B}(\mathcal{H})$ and $C \in \mathcal{B}(\mathcal{K},\mathcal{H})$. Then $\left(\begin{smallmatrix} A & C \\ ? & ? \end{smallmatrix}\right)$ has a quadric-$p(t)$-completion if and only if $R(p(A)) \subseteq R(C)$ and $R(AC) \subseteq R(C)$. Furthermore, X and Y are operators such that $\left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)$ is a quadric-$p(t)$-completion if and only if with respect to the space decomposition $\mathcal{K} = (\ker C)^{\perp} \oplus \ker C$*

$$X = \begin{pmatrix} -C_1^{-1}p(A) \\ X_2 \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} -C_1^{-1}(A + aI)C_1 & 0 \\ Y_{21} & Y_{22} \end{pmatrix},$$

where $C_1 = C\,|_{\ker(C)^{\perp}}$, Y_{22} is any operator on $\ker(C)$ such that $p(Y_{22}) = 0$, and

(i) in the case that $p(t)$ has a double root α,

$$X_2 = \mathrm{s}-\lim_{n\to\infty}(G_2^{(n)} + W_2^{(n)}A - Y_{22}W_2^{(n)} - W_{21}^{(n)}C_1^{-1}p(A))$$

$$Y_{21} = \mathrm{s}-\lim_{n\to\infty}(G_{21}^{(n)} + W_2^{(n)}C_1 - (Y_{22} - \alpha I)W_{21}^{(n)} - W_{21}^{(n)}C_1^{-1}(A - \alpha I)C_1),$$

for some sequences

$$\{G_2^{(n)}\}, \{W_2^{(n)}\} \subseteq \mathcal{B}(\mathcal{H}, \ker C) \text{ and } \{G_{21}^{(n)}\}, \{W_{21}^{(n)}\} \subseteq \mathcal{B}((\ker C)^{\perp}, \ker C)$$

with

$$\mathrm{R}(G_2^{(n)}) \subseteq \ker(Y_{22} - \alpha I), \mathrm{R}(G_{21}^{(n)}) \subseteq \ker(Y_{22} - \alpha I)$$

and

$$(G_2^{(n)} - G_{21}^{(n)}C_1^{-1}(A - \alpha I))\,|_{\overline{\mathrm{R}(A-\alpha I)+\mathrm{R}(C)}} = 0.$$

(ii) in the case that $p(t)$ has two different roots α and β,

$$\begin{aligned} X_2 &= W_2A - W_{21}C_1^{-1}p(A) - Y_{22}W_2 \\ Y_{21} &= W_2C_1 - W_{21}C_1^{-1}(A + aI)C_1 - Y_{22}W_{21} \end{aligned}$$

for some $(W_2\ W_{21}) \in \mathcal{B}(\mathcal{H}\oplus\ker(C)^{\perp}, \ker(C))$.

Proof $\left(\begin{smallmatrix} A & C \\ ? & ? \end{smallmatrix}\right)$ has a quadric-$p(t)$-completion if and only if there exist operators X and Y such that $p(\left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)) = 0$. This holds if and only if $p(A) + CX = 0$, $AC + C(aI + Y) = 0$, $XA + (aI + Y) = 0$ and $XC + p(Y) = 0$.Thus it is trivial to see that if $\left(\begin{smallmatrix} A & C \\ ? & ? \end{smallmatrix}\right)$ has a quadric-$p(t)$-completion, than $\mathrm{R}(p(A)) \subseteq \mathrm{R}(C)$ and $\mathrm{R}(AC) \subseteq \mathrm{R}(C)$.

Conversely, if the condition holds true, then there are operators X and Y_1 such that $p(A) + CX = 0$ and $AC + CY_1 = 0$. Let $Y = Y_1 - aI$, we have $AC + C(Y + aI) = 0$. Notice that $\ker(C) \in \mathrm{Lat}Y$, so, with respect to the space decomposition $\mathcal{K} = \ker(C) \oplus \ker(C)^{\perp}$, we may write $C = (C_1 \ 0)$, $X = \binom{-C_1^{-1}p(A)}{X_2}$ and $Y = \begin{pmatrix} -C_1^{-1}AC_1 - aI & 0 \\ Y_{21} & Y_{22} \end{pmatrix}$. Let $X_2 = 0$, $Y_{21} = 0$ and take any Y_{22} such that $p(Y_{22}) = 0$, it is easily seen that

$$X_0 = \begin{pmatrix} -C_1^{-1}p(A) \\ 0 \end{pmatrix} \text{ and } Y_0 = \begin{pmatrix} -C_1^{-1}(A+aI)C_1 & 0 \\ 0 & Y_{22} \end{pmatrix}$$

make $\begin{pmatrix} A & C \\ X_0 & Y_0 \end{pmatrix}$ a quadric-$p(t)$-completion since $p(Y_{11}) + X_1C_1 = p(-C_1^{-1}(A+aI)C_1) - C_1^{-1}p(A)C_1 = 0$.

Now, let's suppose that $\begin{pmatrix} A & C \\ ? & ? \end{pmatrix}$ has a quadric-$p(t)$-completion and characterize all possible such completions parametrically. We will do this by considering two cases. Under the space decomposition $\mathcal{K} = \ker(C)^{\perp} \oplus \ker(C)$,

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$$

make $\begin{pmatrix} A & C \\ X & Y \end{pmatrix}$ a quadric-$p(t)$-completion if and only if $X_1 = -C_1^{-1}p(A)$, $Y_{11} = -C_1^{-1}(A + aI)C_1$, $Y_{12} = 0$, Y_{22} is any quadric operator with $P(Y_{22}) = 0$ and $E = \begin{pmatrix} X_2 & Y_{21} \end{pmatrix}$ is a solution to the equation

$$Y_{22}E + E\begin{pmatrix} A & C_1 \\ -C_1^{-1}p(A) & -C_1^{-1}(A+aI)C_1 \end{pmatrix} = -aE. \tag{3}$$

(i) The case that $p(t)$ has a double root α, i.e., $p(t) = (t-\alpha)^2$.

Use lemma 2.1, $\begin{pmatrix} X_2 & Y_{21} \end{pmatrix}$ is a solution to (3) if and only if

$$\begin{aligned} \begin{pmatrix} X_2 & Y_{21} \end{pmatrix} &= \mathrm{s}-\lim_{n\to\infty}[\begin{pmatrix} G_2^{(n)} & G_{21}^{(n)} \end{pmatrix} - Y_{22}\begin{pmatrix} W_2^{(n)} & W_{21}^{(n)} \end{pmatrix} + \\ &\qquad \begin{pmatrix} W_2^{(n)} & W_{21}^{(n)} \end{pmatrix}\begin{pmatrix} A & C_1 \\ -C_1^{-1}p(A) & -C_1^{-1}(A+aI)C_1 \end{pmatrix}]. \end{aligned}$$

So,

$$\begin{aligned} X_2 &= \mathrm{s}-\lim_{n\to\infty}(G_2^{(n)} - Y_{22}W_2^{(n)} + W_2^{(n)}A - W_{21}^{(n)}C_1^{-1}p(A)), \\ Y_{21} &= \mathrm{s}-\lim_{n\to\infty}(G_2^{(n)} - Y_{22}W_{21}^{(n)} + W_2^{(n)}C_1 - W_{21}^{(n)}C_1^{-1}(A+aI)C_1) \end{aligned}$$

for some sequence $\left\{\begin{pmatrix} G_2^{(n)} & G_{21}^{(n)} \end{pmatrix}\right\}$ and $\left\{\begin{pmatrix} W_2^{(n)} & W_{21}^{(n)} \end{pmatrix}\right\}$ with

$$(Y_{22} - \alpha I)\begin{pmatrix} G_2^{(n)} & G_{21}^{(n)} \end{pmatrix} = 0$$

and

$$\begin{pmatrix} G_2^{(n)} & G_{21}^{(n)} \end{pmatrix}\begin{pmatrix} A - \alpha I & C_1 \\ -C_1^{-1}p(A) & -C_1^{-1}(A-\alpha I)C_1 \end{pmatrix} = 0.$$

Therefore,

$$\mathrm{R}(G_2^{(n)}) \subseteq \ker(Y_{22} - \alpha I), \mathrm{R}(G_{21}^{(n)}) \subseteq \ker(Y_{22} - \alpha I),$$

$$G_{21}^{(n)}C_1^{-1}(A-\alpha I)\mid_{\overline{\mathrm{R}(A-\alpha I)+\mathrm{R}(C)}}$$

is bounded and $G_2^{(n)} = G_{21}^{(n)}C(A-\alpha I)$ on $\overline{\mathrm{R}(A-\alpha I)+\mathrm{R}(C)}$.

(ii) The case that $p(t)$ has two different roots α and β.

By lemma 2.2, $\begin{pmatrix} X_2 & Y_{21} \end{pmatrix}$ is a solution to (3) if and only if

$$\begin{pmatrix} X_2 & Y_{21} \end{pmatrix} = \begin{pmatrix} W_2 & W_{21} \end{pmatrix}\begin{pmatrix} A & C_1 \\ -C_1^{-1}p(A) & -C_1^{-1}(A+aI)C_1 \end{pmatrix} - Y_{22}\begin{pmatrix} W_2 & W_{21} \end{pmatrix}$$

for some $W \in \mathcal{B}(\mathcal{H}\oplus\ker(C)^{\perp}),\ker(C))$. So,

$$\begin{aligned} X &= W_2A - W_{21}C_1^{-1}p(A) - Y_{22}W_2, \\ Y &= W_2C_1 - W_{21}C_1^{-1}(A+aI)C_1 - Y_{22}W_{21}. \end{aligned}$$

Corollary 2.4 *Assume that $p(t)$, A and C satisfy the hypotheses in Theorem 2.3. Then $\binom{A\ C}{?\ ?}$ has a unique quadric-$p(t)$-completion if and only if $\mathrm{R}(p(A)) \subseteq \mathrm{R}(C), \mathrm{R}(AC) \subseteq \mathrm{R}(C)$ and C is injective.*

Corollary 2.5 *Assume that $p(t)$, A and C satisfy the hypotheses in Theorem 2.3, and assume that $\mathrm{R}(C)$ is closed. Then X and Y make $\binom{A\ C}{X\ Y}$ a quadric-$p(t)$-completion if and only if with respect to decomposition $\mathcal{K} = \ker(C)^{\perp} \oplus \ker(C)$,*

$$X = \begin{pmatrix} -C_1^{-1}p(A) \\ X_2 \end{pmatrix} \text{ and } Y = \begin{pmatrix} -C_1^{-1}(A+aI)C_1 & 0 \\ Y_{21} & Y_{22} \end{pmatrix}.$$

where $C_1 = C\mid_{\ker(C)^{\perp}}$, Y_{22} is any operator on $\ker(C)$ such that $P(Y_{22}) = 0$, and

(i) when $p(t)$ has a double root α,

$$X = \mathrm{s} - \lim_{n\to\infty}(\Psi_n + \Phi_nA - Y_{22}\Phi_n) \text{ and } Y = \mathrm{s} - \lim_{n\to\infty}\Phi_nC_1$$

for some sequences $\{\Phi_n\}$ and $\{\Psi_n\} \subseteq \mathcal{B}(\mathcal{H},\ker(C))$ with $\ker\Psi_n \supseteq \mathrm{R}(A-\alpha I)+\mathrm{R}(C)$ and $\mathrm{R}(\Psi_n) \subseteq \ker(Y_{22}-\alpha I)$;

(ii) when $p(t)$ has two different roots α and β, $X_2 = TA - Y_{22}T$ and $Y_{22} = TC_1$ with $T \in \mathcal{B}(\mathcal{H},\ker(C))$ arbitrary.

Proof (i) If $p(t)$ has a double root α and if X and Y are operators so that $p(\binom{A\ C}{X\ Y}) = 0$, then X and Y must have the forms described in Theorem 2.3. If $\mathrm{R}(C)$ is closed, then C_1^{-1} is bounded. So $G_{21}^{(n)}C_1^{-1}(A-\alpha I) \in \mathcal{B}(\mathcal{H}, \ker(C))$ which is bounded for each n and we write $G_2^{(n)}$ as $G_2^{(n)} = G_{21}^{(n)}C_1^{-1}(A-\alpha I)+ \Psi_n$ with $\ker\Psi_n \supseteq\mathrm{R}(A-\alpha I)+\mathrm{R}(C)$ and $\mathrm{R}(\Psi_n) \subseteq\ker(Y_{22}-\alpha I)$. Let $\Phi_n = G_{21}^{(n)}C_1^{-1} + W_2^{(n)} - Y_{22}W_{21}^{(n)}C_1^{-1} - W_{21}^{(n)}C_1^{-1}(A+aI)$, then $Y = \mathrm{s} - \lim_{n\to\infty}\Phi_nC_1$ and

$$\begin{aligned} X_2 &= \mathrm{s} - \lim_{n\to\infty}[\Psi_n + G_{21}^{(n)}C_1^{-1}(A-\alpha I) + W_2^{(n)}A - Y_{22}W_2^{(n)} - W_{21}^{(n)}C_1^{-1}(A-\alpha I)^2] \\ &= \mathrm{s} - \lim_{n\to\infty}[\Psi_n + \Phi_nA - Y_{22}\Phi_n], \end{aligned}$$

since $p(Y_{22})=0$ and $\mathrm{R}(G_{21}^{(n)})\subseteq\ker(Y_{22}-\alpha I)$.

Conversely, if X and Y satisfy the conditions in the corollary, it is clear that $p(\left(\begin{smallmatrix}A&C\\X&Y\end{smallmatrix}\right))=0$. In fact, let $S_n=\Psi_n+\Phi_nA-Y_{22}\Phi_n$ and $T_n=\Phi_nC_1$, we have

$$(Y_{22}-\alpha I)S_n+S_n(A-\alpha I)-T_nC_1^{-1}p(A)=0,$$

$$(Y_{22}-\alpha I)T_n+S_nC_1-T_nC_1^{-1}(A-\alpha I)C_1=0.$$

Thus $\begin{pmatrix}S_n & T_n\end{pmatrix}$ is a solution to equation

$$(Y_{22}-\alpha I)\begin{pmatrix}S & T\end{pmatrix}+\begin{pmatrix}S & T\end{pmatrix}\begin{pmatrix}A-\alpha I & C_1\\ -C_1^{-1}p(A) & -C_1^{-1}(A-\alpha I)C_1\end{pmatrix}=0$$

for every n and, as the strong limit of $\{\begin{pmatrix}S_n & T_n\end{pmatrix}\}$, $\{\begin{pmatrix}X_2 & Y_{21}\end{pmatrix}\}$ is, too. Therefore, $\left(\begin{smallmatrix}A&C\\X&Y\end{smallmatrix}\right)$ is a quadric-$p(t)$-completion of $\left(\begin{smallmatrix}A&C\\?&?\end{smallmatrix}\right)$.

(ii) $p(t)$ has two different roots α and β.

In this case, if X and Y are operators such that $\left(\begin{smallmatrix}A&C\\X&Y\end{smallmatrix}\right)$ has a quadric-$p(t)$-completion, Theorem 2.3 implies that

$$X=\begin{pmatrix}-C_1^{-1}p(A)\\ X_2\end{pmatrix}\text{ and }Y=\begin{pmatrix}-C_1^{-1}(A+aI)C_1 & 0\\ Y_{21} & Y_{22}\end{pmatrix}$$

with $p(Y_{22})=0$, and

$$\begin{aligned}X_2&=W_2A-W_{21}C_1^{-1}p(A)-Y_{22}W_2,\\ Y_{21}&=W_2C_1-W_{21}C_1^{-1}(A+aI)C_1-Y_{22}W_{21}\end{aligned}$$

for some $\begin{pmatrix}W_2 & W_{21}\end{pmatrix}\in\mathcal{B}(\mathcal{H}\oplus\ker(C)^{\perp},\ker(C))$. Let

$$T=W_2-W_{21}C_1^{-1}(A+aI)-Y_{22}W_{21}C_1^{-1},$$

since $\mathrm{R}(C)$ is closed, T is a bounded operator from $\mathcal{H}$ into $\ker(C)$. Thus $TC_1=Y_{21}$ and

$$TA-Y_{22}T=X_2+p(Y_{22})W_{21}C_1^{-1}=X_2.$$

Conversely, if $X=\left(\begin{smallmatrix}-C_1^{-1}p(A)\\ TA-Y_{22}T\end{smallmatrix}\right)$ and $Y=\left(\begin{smallmatrix}-C_1^{-1}(A+aI)C_1 & 0\\ TC_1 & Y_{22}\end{smallmatrix}\right)$ with Y_{22} satisfying $p(Y_{22})=0$ and $T\in\mathcal{B}(\mathcal{H},\ker(C))$ arbitrary. It is clear that $p(\left(\begin{smallmatrix}A&C\\X&Y\end{smallmatrix}\right))=0$ since

$$(TA-Y_{22}T)(A-\alpha I)-Tp(A)+(Y_{22}-\beta I)(TA-YT)=0,$$

$$(TA-Y_{22}T)C_1-T(A+aI)C_1+(Y_{22}-\beta I)TC_1=0.$$

Corollary 2.6 *Suppose that one of $\mathcal{H}$ and $\mathcal{K}$ is finite dimensional. If $p(t)$, A and C satisfy the hypotheses in Theorem 2.3, then $\left(\begin{smallmatrix}A&C\\X&Y\end{smallmatrix}\right)$ is a quadric-$p(t)$-completion of $\left(\begin{smallmatrix}A&C\\?&?\end{smallmatrix}\right)$ if and only if with respect to the space decomposition $\mathcal{K}=\ker(C)^{\perp}\oplus\ker(C)$,*

$$X=\begin{pmatrix}-C_1^{-1}p(A)\\ X_2\end{pmatrix}\text{ and }Y=\begin{pmatrix}-C_1^{-1}(A+aI)C_1 & 0\\ Y_{21} & Y_{22}\end{pmatrix},$$

where $C_1 = C|_{\ker(C)^\perp}$, Y_{22} is any operator on ker$C$ *such that $p(Y_{22}) = 0$, and*

(i) when $p(t)$ has a double root α,

$$X_2 = \Psi + \Phi A - Y_{22}\Phi \text{ and } Y_{21} = \Phi C_1$$

for some operator Φ and $\Psi \in \mathcal{B}(\mathcal{H}, \ker C)$ with

$$\ker\Psi \supseteq \mathrm{R}(A - \alpha I) + \mathrm{R}(C) \text{ and } \mathrm{R}(\Psi) \subseteq \ker(Y_{22} - \alpha I).$$

(ii) when $p(t)$ has two different roots α and β,

$$X_2 = TA - Y_{22}T \text{ and } Y_{21} = TC_1$$

with $T \in \mathcal{B}(\mathcal{H}, \ker C)$ arbitrary.

Proof It follows immediately from Corollary 2.4 and Lemma 2.1.

3. Norm bounded quadric completion

Now we turn to the discussing the question of norm bounded quadric completion of $\left(\begin{smallmatrix} A & C \\ ? & ? \end{smallmatrix}\right)$. The following lemma is taken from [3] which is needed for our purpose.

Lemma 3.1 $\left(\begin{smallmatrix} A & C \\ ? & B \end{smallmatrix}\right)$ has a completion with norm not greater than $u > 0$ if and only if $AA^* + CC^* \leq u^2 I$ and $B^*B + C^*C \leq u^2 I$. Those X which have the property $\| \left(\begin{smallmatrix} A & C \\ X & B \end{smallmatrix}\right) \| \leq u$ are exactly those of the form $X = -K^*C^*L + u(I - K^*K)^{\frac{1}{2}} Z (I - L^*L)^{\frac{1}{2}}$ with contractions $L = (u^2 I - CC^*)^{-\frac{1}{2}} A$, $K = (u^2 I - C^*C)^{-\frac{1}{2}} B^*$, and with Z an arbitrary contraction.

Using this lemma and Theorem 2.3, we can prove the following result.

Theorem 3.2 *Assume that $p(t)$, A and C satisfy the hypotheses in Theorem 2.3. Then $\left(\begin{smallmatrix} A & C \\ ? & ? \end{smallmatrix}\right)$ has a quadric-$p(t)$-completion $\left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)$ with $\| \left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right) \| \leq u$ for some $u \geq r_p = \max\{|t| : p(t) = 0\}$ if and only if the following inequalities hold:*

$$AA^* + CC^* \leq u^2 I, \tag{4}$$

$$p(A)p(A)^* \leq u^2 CC^*, \tag{5}$$

$$(A + aI)CC^*(u^2 I - CC^*)^{-1}(A + aI)^* \leq CC^*, \tag{6}$$

$$(K^*C^*L - M)(I - L^*L)^{-1}(L^*CK - M^*) \leq u^2(I - K^*K) \tag{7}$$

with $M = C^{-1}p(A)$ and with contractions

$$L = (u^2 I - CC^*)^{-\frac{1}{2}} A, \quad K = -C^{-1}(A + aI)C(u^2 I - C^*C)^{-\frac{1}{2}}.$$

Proof Assume that $p(t)$ has two roots α and β. If X and Y are operators such that $\left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)$ has a quadric-$p(t)$-completion with $\|\left(\begin{smallmatrix} A & C \\ X & Y \end{smallmatrix}\right)\| \leq u$ for some $u \geq r_p$, then $p(A) + CX = 0$, $(A - \alpha I)C + C(Y - \beta I) = 0$, $X(A - \beta I) + (Y - \alpha I)X = 0$ and $XC + p(Y) = 0$. also by use of Lemma 3.1, we have $AA^* + CC^* \leq u^2 I$, $Y^*Y + C^*C \leq u^2 I$, and

$$X = -K^*C^*L + u(I - K^*K)^{\frac{1}{2}} Z (I - L^*L)^{\frac{1}{2}}$$

with contractions $K=(u^2I-C^*C)^{-\frac{1}{2}}Y^*$, $L=(u^2I-CC^*)^{-\frac{1}{2}}A$ and with Z an arbitrary contraction. Hence $p(A)p(A)^*\le u^2CC^*$ since $\| X \|\le u$. Let $C_1=C\mid_{\ker(C)^\perp}$, then with respect to the decomposition $\mathcal{K}=\ker C^\perp\oplus\ker C$,

$$C=\begin{pmatrix} C_1 & 0\end{pmatrix},\quad X=\begin{pmatrix} X_1\\ X_2\end{pmatrix} \text{ and } Y=\begin{pmatrix} Y_{11} & 0\\ Y_{21} & Y_{22}\end{pmatrix}.$$

Clear, $X_1=-C_1^{-1}p(A)$, $Y=-C_1^{-1}(A+aI)C_1$, $CC^*=C_1C_1^*$ and $CC^*=\left(\begin{smallmatrix} C_1^*C_1 & 0\\ 0 & 0\end{smallmatrix}\right)$. So, we have $L=(u^2I-C_1C_1^*)^{-\frac{1}{2}}=L_1$. Because

$$\left\|\begin{pmatrix} A & C_1\\ -C_1^{-1}p(A) & -C_1^{-1}(A+aI)C_1\end{pmatrix}\right\|\le u,$$

we must have

$$C_1^*C_1+C_1^*(A+aI)^*(C_1^{-1})^*C_1^{-1}(A+aI)C_1\le u^2I_1, \tag{8}$$

where $I_1=I\mid_{\ker C^\perp}$ and there exists a contraction $Z_1:\overline{\mathrm{R}(I-L_1^*L_1)}\to\overline{\mathrm{R}(I-K_1^*K_1)}$ such that

$$-C_1^{-1}p(A)=-K_1^*C_1^*L+u(I-K_1^*K_1)^{\frac{1}{2}}Z_1(I-L_1^*L_1)^{\frac{1}{2}}. \tag{9}$$

(8) implies that $C_1^*(A+aI)^*(C_1^*)^{-1}C_1^{-1}(A+aI)C_1\le u^2I_1-C_1^*C_1$, thus there is a contraction K_1 such that $-C_1^{-1}(A+aI)C_1=K_1^*(u^2I_1-C_1^*C_1)^{\frac{1}{2}}$. Therefore, we have

$$\begin{aligned}(A+aI)C_1 &= -C_1K_1^*(u^2I-C_1^*C_1)^{\frac{1}{2}},\\ (A+aI)C_1(u^2I-C_1^*C_1)^{-1}C_1^*(A+aI)^* &= C_1K_1^*K_1C_1^*\le C_1C_1^*,\\ (A+aI)C_1C_1^*(u^2I-C_1C_1^*)^{-1}(A+aI)^* &\le C_1C_1^*.\end{aligned}$$

So

$$(A+aI)CC^*(u^2I-CC^*)^{-1}(A+aI)^*\le CC^*$$

Finally, notice that (9) holds if and only if the operator given by

$$\begin{pmatrix} u(I-K_1^*K_1) & K_1^*C_1^*L-M_1\\ L^*C_1K_1-M_1^* & u(I-L_1^*L_1)\end{pmatrix}$$

is positive, where $M_1=C_1^{-1}(A-\alpha I)^2=C_1^{-1}p(A)$. Put $M=\left(\begin{smallmatrix} M_1\\ 0\end{smallmatrix}\right)$ and notice that

$$K^*=-C^{-1}(A+aI)(u^2I-CC^*)^{-\frac{1}{2}}C=\begin{pmatrix} K_1^* & 0\\ 0 & 0\end{pmatrix},$$

it is easy to see that

$$\begin{pmatrix} u(I-K^*K) & K^*C^*L-M\\ L^*CK-M^* & u(I-L^*L)\end{pmatrix}=\begin{pmatrix} u(I-K_1^*K_1) & 0 & K_1^*C_1^*L-M_1\\ 0 & uI & 0\\ L^*C_1K_1-M_1^* & 0 & u(I-L_1^*L_1)\end{pmatrix}\ge 0$$

and hence $(K^*C^*L-M)(I-L^*L)^{-1}(L^*CK-M^*)\leq u^2(I-K^*K)$. So the conditions are necessary.

Conversely, suppose that the hypotheses hold, let $X=-M$, $Y=K(u^2I-C^*C)^{\frac{1}{2}}$. Obviously, $Y^*Y+C^*C=(u^2I-C^*C)^{\frac{1}{2}}K^*K(u^2I-C^*C)^{\frac{1}{2}}+C^*C\leq u^2I$ and by (4), there exists a contraction $Z:\overline{\mathrm{R}(I-L^*L)}\to\overline{\mathrm{R}(I-K^*K)}$ such that

$$K^*C^*L+X=K^*C^*L-M=u(I-K^*K)^{\frac{1}{2}}Z(I-L^*L)^{\frac{1}{2}}.$$

Hence it follows from Lemma 3.1 that $\|\begin{pmatrix}A&C\\X&Y\end{pmatrix}\|\leq u$. We claim that $\begin{pmatrix}A&C\\X&Y\end{pmatrix}$ is also a quadric-$p(t)$-completion of $\begin{pmatrix}A&C\\?&?\end{pmatrix}$. In fact,

$$\begin{aligned}p(A)+CX&=p(A)-CC^{-1}p(A)=0,\\(A-\alpha I)C+C(Y-\beta I)&=(A-\alpha I)C+C(K(u^2I-C^*C)^{\frac{1}{2}}-\beta I)=0,\\X(A-\beta I)+(Y-\alpha I)X&=-C^{-1}p(A)(A-\beta I)+C^{-1}(A-\beta I)p(A)=0,\end{aligned}$$

and

$$XC+p(Y)=-C^{-1}p(A)C+p(-C^{-1}(A+aI)C)=0.$$

Thus $\begin{pmatrix}A&C\\X&Y\end{pmatrix}$ is a quadric-$p(t)$-completion of $\begin{pmatrix}A&C\\?&?\end{pmatrix}$ with $\|\begin{pmatrix}A&C\\X&Y\end{pmatrix}\|\leq u$. This finishes the proof of the theorem.

Remark 3.3 Under the hypotheses of Theorem 3.2, $\begin{pmatrix}A&C\\X&Y\end{pmatrix}$ is a quadric-$p(t)$-completion of $\begin{pmatrix}A&C\\?&?\end{pmatrix}$ with the norm not greater than $u\geq r_p=\max\{|\alpha|:p(\alpha)=0\}$ if and only if with respect to the space decomposition $\mathcal{K}=\ker C^{\perp}\oplus\ker C$,

$$X=\begin{pmatrix}-C_1^{-1}p(A)\\X_0\end{pmatrix}\text{ and }Y=\begin{pmatrix}-C_1^{-1}(A+aI)&0\\Y_{21}&Y_{22}\end{pmatrix}$$

where $C_1=C|_{\ker C^{\perp}}$, Y_{22} is any operator with $p(Y_{22})=0$ and the norm not greater than u in $\mathcal{B}(\ker C)$, and

$$\begin{pmatrix}X_2&Y_{21}\end{pmatrix}=u(u^2I-Y_{22}Y_{22}^*)^{\frac{1}{2}}Z(u^2I-D^*D)^{\frac{1}{2}}$$

with $D=\begin{pmatrix}A&C_1\\-C_1^{-1}p(A)&-C_1^{-1}(A+aI)C_1\end{pmatrix}$ and Z any contractive solution to the equation

$$(u^2I-Y_{22}Y_{22}^*)^{\frac{1}{2}}Z(u^2I-D^*D)^{\frac{1}{2}}(D-\beta I)+(Y-\alpha I)(u^2I-Y_{22}Y_{22}^*)^{\frac{1}{2}}Z(u^2I-D^*D)^{\frac{1}{2}}=0.$$

Remark 3.4 Let $\mathcal{H}_0\subset\mathcal{H}$ be a subspace and $T_0\in\mathcal{B}(\mathcal{H}_0,\mathcal{H})$. It is interesting to ask when T_0 has an extension T to $\mathcal{H}$ such that, for a given quadric polynomial $p(t)$, $p(T)=0$ or $p(T)=0$ and $\|T\|\leq u$, where $u\geq r_p=\max\{|\alpha|:p(\alpha)=0\}$. Let $\mathcal{H}_1=\mathcal{H}_0^{\perp}$. Then $\mathcal{H}=\mathcal{H}_0\oplus\mathcal{H}_1$ and T_0 can be represented as $T_0=\begin{pmatrix}A&?\\E&?\end{pmatrix}$. So the extension problems becomes the corresponding completion problems. By using Theorem 2.3 and Theorem 3.2, one can easily get the necessary and the sufficient conditions for T_0 to have a quadric-$p(t)$ -extension or quadric-$p(t)$-extension with norm not greater than $u\geq r_p=\max\{|\alpha|:p(\alpha)=0\}$.

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缺项算子矩阵的二阶代数 (I)

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摘 要: 对于任意给定的二阶多项式 $p(t)$, 本文获得希尔伯特空间上形如 $\begin{pmatrix} A & C \\ ? & ? \end{pmatrix}$ 的缺项算子矩阵具有一个补 T 使得 $p(T)=0$ 成立的充分必要条件以及使得 $p(T)=0$ 且 $p(T)$ 的范数不大于事先给定常数的充分必要条件. 进而还求出所有可能的二阶代数补，特别地，对有限维情形给出简洁的表示.