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A Kind of Invariant Hankel Operators *

ZHANG Cheng-guo
(Dept. of Math., Central University for Nationalities, Beijing 100081, China)

Abstract: For two kinds of the Möebius invariant subspace $A_l^{\alpha,2}(D)$ and $\overline{A_l}^{\alpha,2}(D)$ of $L^{\alpha,2}(D)$, we define big and small Hankel operators $H_b^{ll'}$ and $h_b^{ll'}$ for the analytic symbol function $b(z)$, and study their boundedness, compactness and Schatten-von Neumann classes S_p-estimates, and hence develope Schatten-von Neumann properties of these operators.

Key words: weighted Bergman space; Casimir operator; invariant Hankel operator; "periodic" paracommutator; Schatten-von Neumann class.

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1. Introduction

Let D be the unit disk in the complex plane equepped with the Lebesgue measure $\mathrm{dm}(z)$. The Moebius group $G = SU(1,1)$ consists of all 2×2 complex matrices

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a,b,c,d \in C$$

with $c = \bar{b}, d = \bar{a}, ad - bc = 1$. It acts on D via the trasformations

$$z \to gz = g(z) = \frac{az+b}{cz+d}.$$

Let $d\mu_\alpha(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha \mathrm{dm}(z)$ with $\alpha > -1$ and let $L^{\alpha,2}(D)$ be the space consisting of all functions on D square integrable with respect to the measure $d\mu_\alpha(z)$. The group $SU(1,1)$ acts on $L^{\alpha,2}(D)$ via $T_g^\nu : f(z) \to f[g(z)]\{g'(z)\}^{\nu/2} = f(gz)(cz+d)^{-\nu}$, where $\nu = \alpha + 2$.

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Biography: ZHANG Cheng-guo (1945-), male, born in Heishan county, Liaoning province. Currently a professor at central university for Nationalites.

The Casimir operator becomes

$$\Delta_\nu := -4(1-|z|^2)^2\frac{\partial^2}{\partial z\partial\bar{z}} + 4\nu z(1-|z|^2)\frac{\partial}{\partial\bar{z}} - \nu^2 + 2.$$

For $\nu = 0$ i.e. for $\alpha = 2$, the operator Δ_0 has only the continuous spectra and the Plancherel formula has been studied by Harish-Chandra, Helgason et al (see [1]). In this case the space $L^{\alpha,2}(D)$ does not contain any non-trivial analytic function. For $\nu > 1$, the space $L^{\alpha,2}(D)$ contains non-trivial analytic functions, the subspace consisting of all analytic functions is called the (weighted) Bergman space and is denoted by $A_l^{\alpha,2}(D)$. In this case J.Peetre, L.Peng and G.Zhang in [2] and H.Liu and L.Peng in [3] gave the eigenvector of the operator Δ_ν and established the weighted Planchrel formula. They have find that $L^{\alpha,2}(D)$ has some discrete components (invariant subspaces) A_k, where $k < \frac{\alpha+1}{2}$. In other words, the spactra of Δ_ν consist not only of the continuous part, but also of the discrete part and A_k are eigenspaces of Δ_ν with the discrete spectra. They also gave the orthonormal basis of A_k with Romanovski polynomials.

2. Main result

Let P_l and $\overline{P}_l$ denote the orthogonol projections of $L^{\alpha,2}(D)$ onto $A_l^{\alpha,2}(D)$ and $\overline{A}_l^{\alpha,2}(D)$ respectively. Now we define two kinds of Hankel operators for the analytic symbol function $b(z)$: $H_b^{ll'} = P_l M_{\bar{b}} P_{l'}$ and $h_b^{ll'} = \overline{P}_l M_{\bar{b}} P_{l'}$. Because all of the subspaces $A_l^{\alpha,2}(D)$ and $\overline{A}_l^{\alpha,2}(D)$ are invariant under the group actions of $SU(2,R)$, both of the two kinds of Hankel operators are invariant, i.e., we have

$$T_g^\nu H_b^{ll'} = H_{b,g}^{ll'} T_g^\nu, \quad \text{and} \quad T_g^\nu h_b^{ll'} = h_{b,g}^{ll'} T_g^\nu.$$

In this paper we will give the boundedness, compactness and Schatten-von Neumann classes S_p-estimates of them. For S_p classes and analytic Besov spaces B_p^s, may see [4] of the references. The main results of this paper are the following theorems:

Theorem 1 *Let $\alpha > -1, l, l'$ be non-negative integers not excecting $\frac{\alpha+1}{2}$, then for $l > l'$,*

(1) $H_b^{ll'}$ is bounded iff $b \in B_\infty^0$, (Block space);

(2) $H_b^{ll'}$ is compact iff $b \in b_\infty^0$, (Little Bloch space);

(3) if $1 < p < \infty, H_b^{ll'} \in S_p$ iff $b \in B_p^{1/p}$, (Besov space);

(4) if $0 < p \le 1, H_b^{ll'} \in S_p$, then $b = \text{const}$,

for $l = l', H_b^{ll'}$ is bounded iff $b \in L^\infty$, and $H_b^{ll'}$ is never compact unless $b = 0$.

Theorem 2 *Let $\alpha > -1, l, l' < k$, then*

(1) $h_b^{ll'}$ is bouned iff $b \in B_\infty^0$;

(2) $h_b^{ll'}$ is compact iff $b \in b_\infty^0$;

(3) if $0 < p < \infty, h_b^{ll'} \in S_p$ iff $b \in B_p^{1/p}$.

3. The proofs of Theorem 1 and 2

By the orthonormal bases of $A_l^{\alpha,2}(D)$ and $\overline{A}_l^{\alpha,2}(D)$, the two kinds of Hankel operators $H_b^{ll'}$ and $h_b^{ll'}$ are changed into infinite matrices, and then the proofs of Theorem 1 and 2 become the study of the "periodic" paracommutators (see [5],[6],[7]).

The proof of Theorem 1 Notice that $\{e_n^{(l)}(z)=c_{ln}^{-1}p_{ln}(\frac{|z|^2}{1-|z|^2})z^n\}_{n\geq 0}$ is an orthonormal basis of $A_l^{\alpha,2}(D)$ and $\{\bar{e}_n^{(l)}(z)=c_{ln}^{-1}p_{ln}(\frac{|z|^2}{1-|z|^2})\bar{z}^n\}_{n>0}$ is an orthonormal basis of $\overline{A_l}^{\alpha,2}(D)$. For the analytic symbol function $b(z)=\sum_{k=0}^{\infty}\hat{b}(k)z^k$, we now calculate the "matrix coefficient" of $H_b^{ll'}$,

$$\langle H_b^{ll'}(e_n^{(l')}),e_m^{(l)}\rangle=(\alpha+1)\bar{\hat{b}}(n-m)c_l c_{l'}\sqrt{\frac{(\alpha+2-l')_n(\alpha+2-l)_m}{(l'+1)_n(l+1)_m}}\cdot$$

$$\int_0^{\infty}p_{l'n}(t)p_{lm}(t)\frac{t^n}{(1+t)^{n+\alpha+2}}\mathrm{d}t, \tag{1}$$

Using the formula of the hypergeometric function ${}_3F_2$ (see [8]),

$${}_3F_2(-n,a,b;c,1+a+b-c-n;1)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

and

$${}_3F_2(-n,a,b;c,d;1)=\frac{(c-a)_n}{(c)_n}{}_3F_2(-n,a,d-b;1+a-n-c,d;1),$$

we can calculatethe integral of (1)

$$\int_0^{\infty}p_{l'n}(t)p_{lm}(t)\frac{t^n}{(1+t)^{n+\alpha+2}}\mathrm{d}t$$

$$=(n+1)_{l'}(m+1)_l\sum_{\nu=0}^{l}\frac{(-l)_\nu(l-\alpha-1)_\nu(-1)^\nu}{(m+1)_\nu\nu!}\sum_{\mu=0}^{l'}\frac{(-l')_\mu(l'-\alpha-1)_\mu(-1)^\mu}{(n+1)_\mu\mu!}\cdot$$

$$\int_0^{\infty}\frac{t^{n+\nu-\mu}}{(1+t)^{n+\alpha+2}}\mathrm{d}t$$

$$=(m+1)_l(n+1)_{l'}\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\sum_{\nu=l'}^{l}\frac{(-l)_\nu(l-\alpha-1)_\nu(n+1)_\nu}{(m+1)_\nu(-a)_\nu\nu!}\cdot$$

$${}_3F_2(-l',l'-\alpha-1,n+1+\nu;n+1,\nu-\alpha;1)$$

$$=\frac{(-l)_{l'}(l-\alpha-1)_{l'}(m+1)_l(n+1)_{l'}(n+\alpha+2-l')_{l'}}{(-\alpha)_{l'}(-a+l)_{l'}(m+1)_{l'}}\cdot\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}\cdot$$

$${}_3F_2(-l+l',l-\alpha-1+l';n+1+l';m+1+l',-\alpha+2l';1)$$

$$=\frac{(-l)_{l'}(l-\alpha-1)_{l'}(m+1)_l(n+1)_{l'}(n+\alpha+2-l')_{l'}(m-n)_{l-l'}(-m-l)_{l-l'-1}}{(-\alpha)_{2l'}(-a+l')_{l'}(m+1)_{l'}(m+1+l')_{l-l'}(n-m+1-l+l')_{l-l'-1}}\cdot$$

$$\frac{\Gamma(n+1)\Gamma(\alpha+1)}{\Gamma(n+\alpha+2)}{}_3F_2(-l+l'+1,-\alpha+l+l',n+1+l';$$

$$m+2+l',-\alpha+2l';1). \tag{2}$$

Notice that

$$\frac{(m-n)_{l-l'}(-m-l)_{l-l'-1}}{(m+1+l')_{l-l'}(n-m+1-l+l')_{l-l'-1}}\approx\frac{m-n}{m+1+l'},$$

hence we have the estimates $\langle H_b^{ll'}(e_n^{(l')}), e_m^{(l)}\rangle \approx (m+1)^{\frac{\alpha+1}{2}-l-l'}(n+1)^{l'-\frac{\alpha+1}{2}}(n-m)$, where the notation $u \approx \nu$ means that the ratio $\frac{u}{\nu}$ is bounded above and below by constans independent of n and m.

Since $l > l'$, and $l.l' < \frac{\alpha+1}{2}$, we know that $\{\langle H_b^{ll'}(e_n^{(l')}), e_m^{(l)}\rangle\}$ satisfies the condition in [9], by the paracommutator theory, we know that Theorem 1 is true.

The proof of Theorem 2 Similar we only need to calculate the "matrix coefficient" of $h_b^{ll'}$:

$$\langle h_b^{ll'}(e_n^{(l')}), \bar{e}_m^{(l)}\rangle = (\alpha+1)\bar{\bar{b}}(n+m)c_l c_{l'}\sqrt{\frac{(\alpha+2-l')_n(\alpha+2-l)_m}{(l'+1)_n(l+1)_m}}\cdot$$
$$\int_0^\infty p_{l'n}(t)p_{lm}(t)\left(\frac{t}{1+t}\right)^{n+m}\frac{dt}{(1+t)^{\alpha+2}}.$$

Using similar calculate we can obtain the estimates of the "matrix coefficient" of $h_b^{ll'}$:

$$\langle h_b^{ll'}(e_n^{(l')}), \bar{e}_m^{(l)}\rangle \approx (m+1)^{\frac{\alpha+1}{2}-l+l'}(n+1)^{\frac{\alpha+1}{2}-l'}(n+m)^{l-\alpha-1}.$$

By similar reasons we know that Theorem 2 is proved.

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一类不变的 Hankel 算子

张成国

(中央民族大学数学系，北京 100081)

摘 要: 对于 $L^{\alpha,2}(D)$ 的两类 Moebius 不变子空间 $A_l^{\alpha,2}(D)$ 与 $\overline{A_l^{\alpha,2}}(D)$, 定义了对解析的记号函数 $b(z)$ 的大的和小的 Hankel 算子 $H_b^{ll'}$ 与 $h_b^{ll'}$, 研究了它们的有界性、紧性及其 Schatten-von Neumann 类的 S_p 估计. 性质.