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On Graded Essential Right Ideals *

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Abstract: Let R be a graded ring by a group G. For the relative rings $R, R\# G^*$(smash product), $Re, Q(R)$(quotient ring), R^G(fixed ring), $R*G$(crossed product) and normalizing extensions ring S of R, we study the properties of nonsingular, right uniform, right socle. When R_R is a YJ-injective module, we have $J(R)=Z(R)$.

Key words: nonsingular ring; right uniform ring; socle; YJ-injective module.

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1. Introduction

All rings considered in this paper are associative with 1, and all modules are unital. R denotes a graded ring by a group G, and M denotes the graded right R-module.

A ring R is nonsingular provided that $Z(R)=0$, where $Z(R)=\{a\in R|\mathrm{rtann}_R a$ is an essential right ideal of $R\}$ (see [2]). Similarly $Z_G(R)=\{a\in R|$ there is a graded essential right ideal I of R such that $aI=0\}$ is a graded ideal of R. (see [4]). If $Z_G(R)=0$, we say that R is a graded nonsingular ring. It is easy to see that $Z_G(R)\subseteq Z(R)$.

A (graded)ring R is (graded) right uniform provided that each non-zero (graded) right ideal of R is (graded) essential. If R is right uniform ring, then $Z(R)$ consists of all left zero divisor. Let K be a ideal of $R\# G^*$. If $K_e=\{a\in R_e|aP_e\in K\}$, then K_e is an ideal of R_e. The graded radical of a graded module M, denoted by $\mathrm{Rad}_G M$, is the sum of all graded small submodules of M. The socle of the graded module M; denoted by $\mathrm{Soc}_G M$, is the intersection of all graded essential submodule of M, and $\mathrm{Soc}(M)\subset \mathrm{Soc}_G M$.

We say that S is a finite normalizing extension of R if there exists a finite subset $\{a_1,a_2,\cdots,a_n\}$ of S such that $S=\sum\limits_{i=1}^{n}Ra_i$ and $Ra_i=a_iR$ for all $i=1,2,\cdots,n$. It is easy to see that for each a_i, there exists a $\sigma_i\in$ Aut$R$ such that $\sigma_i(r)=r'$, in which $a_ir=r'a_i$, for $r,r'\in R$. If M is a right R-module, then we have $M\bigotimes\limits_R S=\bigoplus\limits_{i=1}^{n}(M\otimes a_i)\cong\bigoplus\limits_{i=1}^{n}M^{\sigma_i}$, where M^{σ_i} is a right R-module $(m\cdot r=mr^{\sigma_i})$.

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Let M be a right R-module. If for each $0 \neq a \in R$, there exists a positive integer n such that $a^n \neq 0$, and for each right R-module morphism from $a^n R$ to M can be extended the morphism from R to M, then M is called a YJ-injetive module. It is easy to see that a right P-injective module is a YJ-injective module. Similarly we can give the definition of graded YJ-injective module.

2. $Z_G(R), Z(R_e)$ and $Z(R\#G^*)$

Theorem 2.1 *Let R be a graded ring by a group G.*

(1) If the grading is nondegenerate, then $Z(R\# G^*)_e = Z(R_e)$ and $Z(R\#G^*) \cap R = Z_G(R)$;

(2) If the grading is faithful, then $Z(R\# G^*) = Z_G(R)\# G^*$;

(3) If G is an ordered group, then $Z_G(R) = Z(R)$.

Proof (1) For any $0 \neq a \in Z(R\# G^*)_e$, then $aP_e \in Z(R\# G^*)$. Let $0 \neq I$ be a right ideal of R_e. Then there exists a $0 \neq x \in \sum\limits_{g\in G} IR_g^{-1}P_g \cap \mathrm{rtann}_{R\#G^*}(aP_e)$. Say $x = \sum\limits_{i=1}^{n} c_i d_i P_{g_0}$, where $c_i \in I, d_i \in R_{g_0}^{-1}$ and $c_i d_i \neq 0$, then $0 \neq \sum\limits_{i=1}^{n}(c_i d_i)R_{g_0} \subseteq I \cap \mathrm{rtann}R_e(a)$, and so $a \in Z(R_e)$. The converse containment is easy to show by [4 Lemma 3 and Theorem 5]. Then $Z(R\# G^*)_e = Z(R_e)$. For $Z(R\# G^*) \cap R = Z_G(R)$, by [4, Lemma 3], we only need to show that if $0 \neq a \in Z(R\# G^*) \cap h(R)$, then $a \in Z_G(R)$. In fact, let $0 \neq I$ be a graded right ideal of R, then exists a $0 \neq b \in I_e, 0 \neq x = \sum\limits_{i=1}^{n} bd_i P_{g_i} \in \sum\limits_{g\in G} bR_g^{-1}P_g \cap \mathrm{rtann}_{R\#G^*}(aP_e)$, where $d_i \in R_{g_i}^{-1}, bd_i \neq 0$ and $abd_i = 0$, so $bd_i \in I \cap \mathrm{rtann}_R(a), a \in Z_G(R)$.

(2) For any $0 \neq x = \sum\limits_{g,h\in G} a_{g,h}P_h \in Z(R\# G^*)$, where $a_{g,h} \in R_g$, there exists a $a_{\alpha,\beta} \neq 0$, so $xP_\beta \neq 0$, and $xP_\beta \in Z(R\#G^*)$. Say that $x = aP_\beta, a \in R$. If I is a nonzero graded right ideal of R, then $I_\beta \neq 0$. Thus, there exists a $0 \neq b \in I_\beta$ and $0 \neq y = \sum\limits_{i=1}^{n} bd_i P_{g_i} \in \sum\limits_{g\in G} bR_g^{-1}P_g \cap \mathrm{rtann}_{R\#G^*}(x)$, where $d_i \in R_{g_i^{-1}}, bd_i \neq 0$. So $xy = 0, abd_i = 0$, and $bd_i \in I \cap (\bigcap\limits_{\sigma\in G} \mathrm{rtann}_R(a_\sigma))$, where $a = \sum\limits_{\sigma\in G} a_\sigma$. It follows that $a \in Z_G(R)$, that is $x \in Z_G(R)\# G^*, Z(R\# G^*) \subseteq Z_G(R)\# G^*$. By [4,Lemma 3], (2) is proved.

(3) For any $0 \neq x \in Z(R)$, suppose that $x = x_{\sigma_1} + x_{\sigma_2} + \cdots + x_{\sigma_n}, (\sigma_1 < \sigma_2 < \cdots < \sigma_n)$. and $x_{\sigma_i} \neq 0$. then $x_{\sigma_n} \in Z(R)^\sim$. Since there is an essential right ideal I of R such that $xI = 0$, so $x_{\sigma_n}I^\sim = 0$. and $I^\sim$ is a graded essential right ideal of R. We have $x_{\sigma_n} \in Z_G(R) \subseteq Z(R)$, and $x - x_{\sigma_n} \in Z(R)$. An argument such as above by replace x to $x - x_{\sigma_n}$, we have $x_{\sigma_i} \in Z_G(R), i = 1, 2, \cdots, n$. Hence $x \in Z_G(R)$. that is $Z(R) \subseteq Z_G(R)$.

Theorem 2.2 *Let G be a finite group, G act on a ring A, Then*

(1) $Z(A * G) \cap A = Z(A)$;

(2) If A is semiprime and A has no $|G|$-torsion. Then $Z(A^G) = Z(A)^G$ and $Z(A)*G = Z(A * G)$.

Proof (1) Let $0 \neq r \in Z(A)$, I be a non-zero right ideal of $A*G$. If $0 \neq s = \sum\limits_{i=1}^{n} r_i\overline{\sigma}_i \in I$, and

$r_k \notin \mathrm{rtann}_R(r)$ for some integer k. Then there exists a $0 \neq x \in A$, such that $r_k x \neq 0$, and $rr_k x = 0$. Say that $x' \in A$ such that $\overline{\sigma}_k x' = x\overline{\sigma}_k$, then $0 \neq sx' \in I$ and $r_k x \in \mathrm{rtann}_A(r)$, where $r_k x$ is the coefficient of $\overline{\sigma}_k$ within sx'. It follows that sx' has more confficients in $\mathrm{rtann}_A(r)$, than s does. Thus there exists a $s \in I$ such that all the confficients of s belong to $\mathrm{rtann}_A(r)$, and so $r \in Z(A*G), Z(A) \subseteq Z(A*G)$. Conversely, let $0 \neq r \in Z(A*G) \cap A$, I be a non-zero right ideal of A. Then we have $0 \neq x = \sum_{i=1}^{n} a_i\overline{\sigma}_i \in I(A*G) \cap \mathrm{rtann}_{A*G}(r)$, where $a_i \in I$. So $ra_i = 0$ and $a_i \in I \cap \mathrm{rtann}_A(r), r \in Z(A)$.

(2) Let $0 \neq a \in Z(A)^G$ and I be a non-zero right ideal of A^G. Then IA is a G-invariant right ideal of A. Say $J = \mathrm{rtann}_A(a)$, so $(\bigcap_{g\in G} J^g) \cap IA \neq 0$. By Theorem 4.3 of [7], we have $I \cap J \supseteq \mathrm{tr}((\bigcap_{g\in G} J^g) \cap IA) \neq 0$ and $a \in Z(A^G)$. The equality $Z(A)^G = Z(A^G)$ will follow if we apply the Lemma 5.7 of [7]. Beeause $A*G$ is a strongly graded ring, and $(A*G)\# G^*$ is a semiprime ring. By theorem 2.1 and Lemma 3 of [4], it is easy to show that $Z(A)*G = Z(A*G)$.

Theorem 2.3 *Let S be the set of all regular elements of A; $Q(A)$ be the classic quotient ring. Then $Z(Q(A)) = Z(A)S^{-1}$.*

Proof For each $0 \neq x \in Z(Q(A))$, say $x = ab^{-1}(a \in A, b \in S)$. Then $\mathrm{rtann}_A(x)$ be an essential right ideal of A by $\mathrm{rtann}_{Q(A)}(x)$ is an essential right ideal of $Q(A)$. If J is a non-zero right ideal of A, then there exists a $0 \neq d \in J$ such that $0 \neq bd \in \mathrm{rtann}_A(x)$. and $ad = 0, J \cap \mathrm{rtann}_A(a) \neq 0, a \in Z(A)$. Thus $Z(Q(A)) \subseteq Z(A)S^{-1}$. Conversely, let $0 \neq x = ab^{-1} \in Z(A)S^{-1}$, where $a \in Z(A), b \in S, J$ be a right ideal of $Q(A)$. Then $(J \cap A) \cap \mathrm{rtann}_A(a) \neq 0, J \cap \mathrm{rtann}_{Q(A)}(a) \neq 0, a \in Z(Q(A))$. Thus $Z(A)S^{-1} \subseteq Z(Q(A))$.

3. Right uniform rings

Theorem 3.1 *Let S be a finite normalizing extension of ring R. Then S is a right uniform ring if and only if R is a right uniform ring.*

Proof If K is a non-zero right ideal of R, then KS is an essential right ideal of S. By Proposition 1.1 of [5], KS_R is an essential submodule of S_R. Since $KS \cong \bigoplus_{i=1}^{n} K^{\sigma_i}$, and $S \cong \bigoplus_{i=1}^{n} R^{\sigma_i}$, thus K_R is an essential submodule of R_R. Conversely, let K be a nonzero right ideal of S, and $K_i = \{r \in R |$ the re are $r_1, r_2, \cdots, r_{i-1}, r_{i+1}, \cdots, r_n \in R$ such that $r_1a_1 + r_2a_2 + \cdots + r_{i-1}a_{i-1} + ra_i + r_{i+1}a_{i+1} + \cdots + r_na_n \in K\}$. Then K_i is a nonzero right ideal of R. So K_i is essential. Say that $J = \bigcap_{j=1}^{n} K_i$. Then J is an essential right ideal of R. For any nonzero right ideal T of S, $0 \neq I = \bigcap_{i=1}^{n} T_i$ is a right ideal of R(Here T_i is similar to K_i above. $i = 1, 2, \ldots, n$). Thus $0 \neq I \cap J \subseteq (I \cap J)S \subseteq IS \cap JS \subseteq K \cap T$, and K is essential.

Theorem 3.2 *Let G be a finite group, and G act on a ring A. For the following statements: (1) $A*G$ is a right unform ring; (2) A is a right uniform ring; (3) A is a right*

G-uniform ring and (4)A^G is a right uniform ring; we have

(I) (1)$\Leftrightarrow$ (2) $\Rightarrow$ (3).

(II) If A is semiprime and A has no $|G|$-torsion, then (3)$\Leftrightarrow$ (4).

Proof By Theorem 3.1, (1)$\Leftrightarrow$ (2) and (2)$\Rightarrow$ (3) are easy. (4) $\Rightarrow$ (3) follows from [7, Lemma 5.1]. Now we show (3) $\Rightarrow$ (4). Let K be a nonzero right ideal of A^G. Then KA is a G-invavaint right ideal of A, and so is G-essential. Then K is an essential right ideal of A^G.

Theorem 3.3 *Let R be a G-graded ring. For the following statements: (1)$R\#\ G^*$ is a right G-uniform ring; (2) R is a graded right uniform ring; (3) R is a right uniform ring and (4) R_e is a right uniform ring; we have*

(I) (1)$\Rightarrow$ (2) $\Leftarrow$ (3), (3) $\Rightarrow$ (1).

(II) If R is a commutative ring, then (1)$\Leftrightarrow$ (2).

(III) If the grading is nondegenerate, then (2)$\Leftrightarrow$ (4).

Proof (1) $\Rightarrow$ (2) Let I_1, I_2 be two graded right ideals of R. Then $I_i(R\#\ G^*)$ is a G-invariant right ideal of $R\#\ G^*(i = 1,2)$. So $I_1(R\#\ G^*) \cap I_2(R\#\ G^*) \neq 0$. Similar to the proof of theroem 2.1(2), there exists a $\beta \in G$ and $a \in R$ such that $0 \neq ap_\beta \in I_1(R\#\ G^*) \cap I_2(R\#\ G^*)$. So we have $0 \neq a \in I_1 \cap I_2$. Hence I_1 is a graded essential right ideal.

(3)$\Rightarrow$ (1) Let I be a nonzero G-invariant right ideal of $R\#G^*$. Then there exist $0 \neq a \in R$ and $\beta \in G$ such that $ap_\beta \in I$, and $ap_g = (ap_\beta)^{\beta^{-1}g} \in I$, for all $g \in G$. Hene $a \in I \cap R.aR$ is essential. Let J be a non-zero G-invariant right ideal of $R\#\ G$, then trere exists a $0 \neq b \in R$ such that $b \in J \cap R$, it follows that $0 \neq aR \cap bR \subseteq I \cap J$, that is I is G-essential.

The others are easy.

Theorem 3.4 *$Q(A)$ is a right unform ring if and only if A is a right uniform ring.*

Proof Let I, J are non-zero right ideals of R. Then IS^{-1} is an essential ideal of $Q(A)$, and $IS^{-1} \cap A$ is an essential right ideal of A. Thus $(IS^{-1} \cap A) \cap J \neq 0$ and $I \cap J \neq 0$, that is I is an essential right ideal. Conversely, let k be a non-zero right ideal of $Q(A)$. Then $K = (K \cap A)S^{-1}$, so $(K \cap A)S^{-1}$ is an essential right ideal of $Q(A)$.

4. Socle

Theorem 4.1 *Let G be a finite group, G act on a ring A. Then*

(1) $\mathrm{Soc}A \supseteq A \cap \mathrm{Soc}(A * G)$.

(2) If A is semiprime and A has no $|G|$-torsion, then $\mathrm{Soc}A = A \cap \mathrm{Soc}(A * G), \mathrm{Soc}(A * G) = (\mathrm{Soc}\ A) * G$.

Proof (1) For any $r \in A \cap \mathrm{Soc}(A * G)$. we have $r(A * G)$ is a completely reducible $A * G$-module. By [6, Theorem 4], $r(A * G)$ is a completely reducible A-module. Since rA is a A-submodule of $r(A * G)$, so rA is a completely reducible module and $r \in \mathrm{Soc}(A)$.

(2) Let $r \in \mathrm{Soc}A$. Then rA_A is completely reducible. Since $r(A * G) = \sum_{i=1}^{n} rA\overline{\sigma}_i$, so $r(A * G)$ is a completely reducible right A-module. Thus $r(A * G)$ has a finite composition

series as a $A*G$-module. It follows that $A*G$ is semiprime and each minimum ideal of $A*G$ generate by idempotent elements. So that $r(A*G)$ is completely reducible right $A*G$-module, hence $r\in \mathrm{Soc}(A*G)$. Conversely, for any $x\in \mathrm{Soc}(A*G),(x*(A*G))_{A*G}$ is a completely reducible module, and $x(A*G)$ is also a completely reducible right A-module. Thus xA is a completely reduible right A-module. Let $xA=\sum_{j=1}^{m} y_jA$ be a sum of simple A-module. For each y_i, say that $y_i=\sum_{i=1}^{n} r_{ji}\overline{\sigma}_i$. We have a right A-module morphism $f_{ji}: y_jA\to r_{ji}A, y_ia\to r_{ji}a$. Since $A*G$ is a free normalizing extension then the above map is well defined. If $r_{ji}\neq 0$. then $y_iA\cong r_{ji}A$, and $r_{ji}A$ is a simple A-module. So $r_{ji}\in \mathrm{Soc}A, y_iA=\sum_{i=1}^{n} r_{ji}A\overline{\sigma}_i\subseteq(\mathrm{Soc}A)*G, x\in(\mathrm{Soc}(A))*G$.

Theorem 4.2 *Let R be a G-graded ring. Then*

(1) If G is a finite group and R is a commutative ring, then $\mathrm{Soc}_G(R)\ \#\ G^*=\mathrm{Soc}(R\#\ G^*)$.

(2) If R_e is a simple Artin ring and R_e has no $|G|$-torsion. Supp $R=\{x\in G|R_x\neq 0\}=G$, *then* $\mathrm{Soc}_G(R)=\mathrm{Soc}R$.

Proof (1) Let J be an essential right ideal of $R\#\ G^*$. Then $\bigcap_{g\in G} J^g$ is an essential right ideal of $R\#\ G^*$. Suppose I is a graded essential right ideal of R, then $(\bigcap_{g\in G} J^g)\cap I(R\#\ G^*)\neq 0$, An argument similar to the Theorem 3.3, we have that there exists a $0\neq a\in R$ such that $aP_e\in(\bigcap_{g\in G} J^g)\cap I(R\#\ G^*)$. So $a\in I$ and $aP_e\in\bigcap_{g\in G} J^g$. Hence $ap_h=(ap_e)^h\in\bigcap_{g\in G} J^g,\forall h\in G$. It follows that $a=\sum_{h\in G} aP_h\in\bigcap_{g\in G} J^g, 0\neq a\in I\cap((\bigcap_{g\in G} J^g)\cap R)$.$\mathrm{Soc}_G R\subseteq(\bigcap_{g\in G} J^g)\cap R\subseteq J, \mathrm{Soc}_G(R)\subseteq \mathrm{Soc}(R\#\ G^*)$. Conversely, let I be a graded essential right ideal of R, J be a non-zero right ideal of $R\#\ G^*$, there exists $a\beta\in G, 0\neq a\in R$ such that $ap_e\in J^{\beta^{-1}}, I\cap aR\neq 0$. Hence $0\neq(I\cap aR)P_\beta=((I\cap aR)P_e)^\beta\subseteq IP_\beta\cap J\subseteq I(R\#\ G^*)\cap J$. Thus $I(R\#\ G^*)$ is an essential right ideal of $R\#\ G^*$.$\mathrm{Soc}(R\#\ G^*)\subseteq I(R\#\ G^*)$. For any $x=\sum_{g\in G} a^gP_g\in \mathrm{Soc}(R\#\ G^*)$, where $a^g\in R$. Since $\mathrm{Soc}(R\#\ G^*)$ is G-invaria nt, for every g, we have $a^gP_g=(xP_g)^{g^{-1}\beta}\in \mathrm{Soc}(R\#\ G^*)\subseteq I(R\#\ G^*),(\forall\beta\in G)$ and $a^g\in I(\forall I)$. Thus $a^g\in \mathrm{Soc}_G R(\forall g\in G)$, and so $x\in(\mathrm{Soc}_G R)\#\ G^*$.

(2) By [8, proposition 1.11], we have $R=R_e*G$.$\mathrm{Soc}R=(\mathrm{Soc}R_e)*G$ by the Theorem 4.1.(2). Since R_e*G is a strongly graded ring, by the theorem 4.2(1), we have $\mathrm{Soc}_G(R_e*G)=\mathrm{Soc}(R_e)\cdot(R_e*G)=(\mathrm{Soc}R_e)*G$. That is $\mathrm{Soc}R=\mathrm{Soc}_G R$.

For the classical quotient ring, we have

Theorem 4.3 *Let $Q(A)$ be a right quotient ring of A, S be the set of all regular elements of A. Then* $(\mathrm{Soc}A)S^{-1}=\mathrm{Soc}(Q(A))$.

5. Graded radical

In this section. G is an ordered group.

Lemma 5.1 *Let $m \in \mathrm{Rad}M$. Then $m_{\sigma_n}R$ is a graded small submodule and hence is small submodule, where m_{σ_n} is the highest degree homogeneous component of m.*

Proof Suppose $m_{\sigma_i}R$ is not a graded small submodule. Then $\sum = \{B$ is a graded submodule of $M | B \neq M$ and $m_{\sigma_n}R + B = M\}$. We may apply Zorn's Lemma to $\sum$ and choose a maximal such graded submodule, call it C. There exists a maximal submodule K of M such that $K^{\sim} = C$. Then $m \notin K$. But mR is a small submodule, a contradiction. It follows that m_{σ_n} is a graded small submodule. Let K be a submodule of M, and $m_{\sigma_n}R + K = M$. Then $mR + K = M, K = M$. So $m_{\sigma_n}R$ is a small submodule.

Lemma 5.2 $(\mathrm{Rad}M) \sim = (\mathrm{Rad}M)_G = \mathrm{Rad}M \cap \mathrm{Rad}_G M$.

Proof For any $x \in (\mathrm{Rad}M)^{\sim} \cap h(M)$, there exists a $m \in \mathrm{Rad}M$, such that $m_{\sigma_n} = x$. By Lemma 5.1, we have $x \in \mathrm{Rad}_G M \cap \mathrm{Rad}M \subseteq (\mathrm{Rad}\ M)_G$, and $(\mathrm{Rad}M)^{\sim} \subseteq \mathrm{Rad}M \cap \mathrm{Rad}_G M \subseteq (\mathrm{Rad}M)_G \subseteq (\mathrm{Rad}M)^{\sim}$.

Theorem 5.3 $(\mathrm{Rad}M)_G = (\mathrm{Rad}M)^{\sim} = \mathrm{Rad}M \subseteq \mathrm{Rad}_G M$.

Proof For any $m \in \mathrm{Rad}M$, we have $m_{\sigma_n} \in (\mathrm{Rad}M)^{\sim} \subseteq \mathrm{Rad}M \cap \mathrm{Rad}_G M$. Similar to the proof of Theorem 2.1(3), we get $m \in \mathrm{Rad}_G M \cap (\mathrm{Rad}M)^{\sim}$. That is $\mathrm{Rad}M \subseteq \mathrm{Rad}_G M \cap (\mathrm{Rad}M)^{\sim}$.

Theorem 5.4 *Let $R/\mathrm{Rad}_G R$ be a semisimple ring. Then* $\mathrm{Soc}_G R = \mathrm{Soc}R$.

Proof Since $R/\mathrm{Rad}_G R$ is a semisimple ring, so $R/\mathrm{Rad}_G R$ is graded seimisimple. It is easy to show that $\mathrm{Soc}_G R = lt\mathrm{ann}_R(\mathrm{Rad}_G R)$. And $lt\mathrm{ann}_R(\mathrm{Rad}_G R)$ is a semisimple right $R/\mathrm{Rad}_G R$-module. So $lt\mathrm{ann}_R(\mathrm{Rad}_G R)$ is a semisimple right R-module. $lt\mathrm{ann}_R(\mathrm{Rad}_G R) \subseteq \mathrm{Soc}R$. Thus $\mathrm{Soc}_G R \subseteq \mathrm{Soc}R$. Convesely, it is easy to see.

Theorem 5.5 *Let G be a finite group. R be a G-graded ring. Then* $\mathrm{Rad}(R\#\ G^*) = (\mathrm{Rad}_G R)\#\ G^*$.

Proof First, let I be a maximal graded right ideal of R. Say that $\sum = \{J$ is right ideal of $R\#\ G^* | J \cap R = I\}$. Since G is a finite group, then $I = I(R\#\ G) \cap R$, and so $\sum$ is nonempty. We may apply Zorn's lemme to $\sum$ and choose a maximal such right ideal, call it Q. Then $\mathrm{Rad}(R\#\ G^*) \cap R \subseteq Q \cap R = I, \mathrm{Rad}(R\#\ G^*) \cap R \subseteq \mathrm{Rad}_G R$. For any $x \in \mathrm{Rad}(R\# G^*)$, let $x = \sum\limits_{g \in G} a^g P_g$. Then $xP_g = a^g P_g \in \mathrm{Rad}(R\#\ G^*)$ for each $g \in G$. It follows that $a^g P_\beta = (a^g P_g)^{g^{-1}\beta} \in (\mathrm{Rad}(R\#\ G^*))^{g^{-1}\beta} \subseteq \mathrm{Rad}(R\#\ G^*), \forall \beta \in G$. and so $a^g \in \mathrm{Rad}(R\# G^*) \cap R \subseteq \mathrm{Rad}_G R.(\forall g \in G)$ Thus, $x \in (\mathrm{Rad}_G R)\sharp G^*$, and $\mathrm{Rad}(R\#\ G^*) \subseteq (\mathrm{Rad}_G R)\#\ G^*$. Conversely, let I be a maximal G-right ideal of $R\#\ G^*$, say that $\sum = \{J$ is a right ideal of $R\#\ G^* | \bigcap\limits_{g \in G} J^g = I\}$. It is easy to see that there exists a maximal element Q in $\sum$. Since $I \cap R$ is a maximal graded right ideal, so $(\mathrm{Rad}_G R)\#\ G^* \subseteq (I \cap R)\#\ G^* \subseteq I = \bigcap\limits_{g \in G} Q^g$. Thus $(\mathrm{Rad}_G R)\#\ G^* \subseteq \mathrm{Rad}(R\# G^*)$.

6. YJ-injective module

Theorem 6.1 *Let M be a YJ-injective right R-module. Then M is a divisible module.*

Proof For any $m_0 \in M, r_0 \in R$ and r_0 is not a left zero divisor, there exists a positive integer n such that $r_0^n \neq 0$ and the right R-module morphism from $r_0^n R$ to M can be extended to a right R-module morphism from R to M. Define $f : r_0^n R \to M, r_0^n r \to m_0 r$. It is easy to see that f is a right R-module morphism. There exists an $x \in M$ such that $m_0 = f(r_0^n) = x r_0^n = (x r_0^{n-1}) r_0 (m_0 = x r_0$ When $n = 1)$. Thus M is a di visible module.

Theorem 6.2 *Let S be an Excellent extension of R, M_S be a YJ-in jective module. Then M_R is a YJ-injective module.*

Proof For any $a \in R$, thexe exists a positive integer n such that $a^n \neq 0$, and the right R-module morphism from $a^n S$ to M can be extended to right R-module morphism from S to M. Let $f : a^n R \to M$ be a right R-module morphism. Say that $J = a^n R_{a_1} + a^n R_{a_2} + \cdots + a^n R_{a_n} = a^n S$, Suppose $F : J \to M, \sum_{j=1}^{n} x_i a_i \to \sum_{i=1}^{n} f(x_i) a_i$ (where $x_i \in a^n R$.) Then F is well defined. This follows F is a right S-module morphism. And there exists a morphism $G : S \to M$ such that $G|_J = F$, so $G|_R : R \to M$ is a right R-module morphism and $(G|_R)|_{a_n R} = f$.

Theorem 6.3 *Let R_R be a YJ-injective module, Then $J(R) = Z(R)$.*

Proof For any $b \in Z(R)$ and $a \in R$, we have $\mathrm{rtann}_R(ab) \cap rt(\mathrm{ann}_R(1 - ab)) = 0$. Since $ab \in Z(R)$, so $\mathrm{rtann}_R(1 - ab) = 0$. There is a positive integer n such that for any right R-module morphism from $(1 - ab)^n R$ to R can be extented right R-module morphism form R to R. Set $f : (1 - ab)^n R \to R, (1 - ab)^n r \to r$, then f is a right R-module morphism. So there exists a $y \in R$ such that $1 = f((1 - ab)^n) = y(1 - ab)^n = y((1 - ab)^{n-1})(1 - ab)$, that is ab is a right quasi-regular element for any $a \in R$. Thus $b \in J(R)$ and $Z(R) \subseteq J(R)$.

Conversely, Let $b \in J(R)$ but $b \notin Z(R)$. There is a non-zero right ideal I of R such that $\mathrm{rtann}_R(b) \bigoplus I$ is an essential right ideal of R. For $0 \neq c \in I$, there exists a positive integer n such that $(bc)^n \neq 0$. Set $g : (bc)^n R \to R, (bc)^n r \to c(bc)^{n-1} r (bcr \to cr$ when $n = 1)$. It is easy to see that g is a right R-module morphism. There is $ad \in R$ such that $c(ab)^{n-1} = g((bc)^n) = d(bc)^n = dbc(bc)^{n-1}$. Since $db \in J(R)$, so there is a $k \in R$ such that $db + k - kbd = 0$. Then $c(bc)^{n-1} = dbc(bc)^{n-1} = kdbc(bc)^{n-1} - kc(bc)^{n-1} = 0$. Acontradiction. Thus $b \in Z(R)$.

By the Theorem 6.1 and 6.3, we have

Theorem 6.4 *Let R be a right uniform right and R_R be a YJ-injective modulee Then*

(1) $J(R)$ consists of all zero divisors.
(2) $R/J(R)$ is a dvisible ring.
(3) R is a local ring.
(4) R is an IBN ring.

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关于分次本质右理想

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摘 要: 设 R 是 G- 分次，本文讨论了环 R 的相关环 $R, R\# G^*, Re, Q(R), R^G, R*G$ 及 R 的正规化扩张 S 的非奇异性，右一致性，右基座之间的关系. 当 R_R 是 YJ- 内射模时，证明了 $J(R) = Z(R)$.