Journal of Mathematical Research & Exposition
Vol.20, No.4, 475–482, November, 2000

Monadicity of Inj_0 over Top *

ZHAO Dong-sheng[1], *ZHAO Bin*[2]

(1. Nanyang Technological University, Singapore;

2. Dept. of Math., Shaanxi Normal University, Xi'an 710062, China)

Abstract: In this paper, we mainly prove that the category Inj_0 of all injective T_0-spaces and strongly algebraic maps is monadic over Top by showing that Inj_0 is equal to the Eilenberg-Moore category Top^T, where T is the monad produced by an dual adjunction between the category Top and the category Slat of all meet-semilattices which have top elements and semilattice homomorphisms.

Key words: category; dual adjunction; meet-semilattice.

Classification: AMS(1991) 06B35,06A12/CLC O153.1

Document code: A **Article ID:** 1000-341X(20000)04-0475-08

1. Introduction

Since Stone first gave the topological representations for Boolean algebras in 1930's there have been massive study on adjunctions between the category Top of topological spaces and some concrete category C. In many cases the functor F:Top $\to$ C^{op} is defined in such a way that for any topological space X the underlying set of $F(X)$ is $O(X)$-the set of all open sets of X, and for each continuous map $f: X \to Y, F(f) = f^{-1}: O(Y) \to O(X)$. The right adjoint of F is usually defined as Spec:C $\to$ Top, where for each object B of C Spec$B$=C$(B,2)$ (where 2 is an object of C whose underlying set is the two elements set) with the subspace topology of the product space 2^B. This adjunction produces a monad $T = (T, \eta, \mu)$ on Top and a monad $R = (R, \varepsilon, \nu)$ on C. An important problem is to characterize the Eilenberg-Moore categories Top^T and Set^R. For C=Set Hoffmann ([1]) showed that Set^R is (up to isomorphism) the category Frm of frames (see [2] for the definition of Frm) and Sobrel showed that Top^T is the category LInj-T_0([3]). In [4] Simmons carefully studied the case for C=DLat, the category of all distributive lattices. He showed that the category Top^T is the category of all algebraic spaces and algebraic maps, and that DLat^R is Frm.

*Received date: 1998-06-25
Foundation item: Supported by the Natural Science Foundation of China (19701020) and Supported by the Teaching and Research Award Fund for Qutstanding Young Teachers in Higher Education Institiions of MOE.

The main purpose of this paper is to describe the category Top^T and Slat^R for the category Slat of all meet-semilattices. It will be shown that Top^T is (up to isomorphism) the category Inj_0 of all injective T_0-spaces and strongly algebraic maps.

2.Basic structures and facts

By a concrete category we mean a category C whose objects are structured sets, i.e. pairs (X,ξ) where X is a set and ξ is a C -structure on X, whose morphism $f:(X,\xi)\to(Y,\eta)$ are suitable maps between X and Y and whose composition law is the usual composition of maps. In other words: a concrete category is a category C together with a faithful functor F:C $\to$ Set. A concrete functor F:C $\to$ D between two concrete categories C and D is a functor such that $F((X,\xi))$ has the underlying set X for each $(X,\xi)\in$ C. We use $|X|$ to denote the underlying set of an object X of C.

A frame is a complete lattice which satisfies the infinite distributive law:

$$a\wedge\bigvee_{i\in I}x_i=\bigvee_{i\in I}(a\wedge x_i).$$

A frame homomorphism is a map which preserves finite meets and arbitrary joins. Let Frm denote the category whose objects are frames and whose morphisms are frame homomorphisms. The category Frm is obviously a concrete category. We need the following condition for the concrete category C we will deal with:

There exists a faithful concrete functor $\varepsilon:\text{Frm}\to C.$ $(*)$

If C satisfies $(*)$, by abusing language we use the same symbol A to denote $\varepsilon(A)$ for each frame A, and use the same symbol $f:A\to B$ to denote $\varepsilon(f)$ for each frame morphism $f:A\to B$ because these do not cause any confusion. Then clearly we have a functor O:Top $\to \text{C}^{op}$ which sends each topological space X to its open sets frame $O(X)$ and each continuous map $f:X\to Y$ to $O(f)=f^{-1}:O(Y)\to O(X)$.

We now define a functor Spec:$\text{C}^{op}\to$ Top. We use 2 to denote the two elements chain when we regard it as a frame and use 2 to denote the Sierpinski space when we regard it as a topological space. For each object B of C, $\text{Spec}(B)=\text{C}(B,2)$, whose topology has a subbase $\{\sigma(c)\mid c\in|B|\}$ where $\sigma(c)=\{f\mid f\in\text{Spec}(B),f(c)=1\}$. For C morphism $h:B\to D,\text{Spec}(h):\text{Spec}(D)\to\text{Spec}(B)$ is the map which sends $f\in\text{C}(D,2)$ to $f\circ h\in\text{C}(B,2)$. Since $\text{Spec}(h)^{-1}(\sigma(c))=\sigma(h(c))$ for each $c\in|B|$, so Spec(h) is a continuous map. It is clear that Spec is a functor. If $B\in$C then there is a map $\varepsilon_B:|B|\to|O(\text{Spec}(B))|$ which sends $c\in|B|$ to $\sigma(c)$.

Now we need another extra condition on C:

$\varepsilon_B:B\to O$ (Spec(B)) is a C morphism for each $B\in$ C. $(**)$

It is straightforward to show that for each C object B, $\varepsilon_B:B\to O(\text{Spec}(B))$ is the universal morphism from B to the functor O (regarded as a functor from TOP^{op} to C).

If X is a topological space, there is a continuous map $\eta_X:X\to\text{Spec}(O(X))$ which sends $y\in X$ to $\eta_X(y)$ such that for each $U\in O(X),\eta_X(y)(U)=1$ iff $y\in U.\eta_X(y)$ is obviously a frame morphism so it is really in Spec$(O(X))$. From the condition $(*)$ it follows that η_X is a universal map from X to the functor Spec.

Combining all the above arguments we get the following lemma.

Lemma 2.1 *Let C be a concrete category satisfying the conditions (*) and (**), then the functors O:Top $\to C^{op}$ and Spec:$C \to$ Top^{op} are dually adjoint to each other.*

This dual adjunction produces a monad $T = (T, \eta, \mu)$ on Top and a monad $R = (R, \varepsilon, \upsilon)$ on C.

Examples 2.2 (1)The category C=Set of all sets is obviously a concrete category satisfying the conditions (*) and (**). In [3] Sobrel proved that in this case Top^T is, the L-subcategory of Inj-T_0 (the full subcategory of Top whose objects are all injective T_0-spaces), while Hoffmann characterized Set^R as the category Frm.

(2) The category Dlat of all distributive lattices and lattice homomorphisms is a concrete category satisfying conditions (*) and (**). Simmons proved that Top^T is, up to isomorphism, the category AlgSpac of all algebraic spaces and algebraic maps and Dlat^R is again Frm (see[2]).

(3) We can also take Frm as C. In this case, by a direct verification it can be proved that Top^T is isomorphic to the category Sober of all sober spaces and continuous maps (just notice that a retract of a sober space is sober), and Frm^R is Frm itself.

On any complete lattice L there is an relation $\lhd$ which is defined as follows:$a \lhd b$ if and only if for each set B, if $\vee B \geq b$ then $a \leq x$ for some $x \in B$. An element a of a complete lattice L is called supercompact if $a \lhd a$. A complete lattice is called supercontinuous if for each $a \in L$, $a = \vee\{x \in L \mid x \lhd a\}$.

Remark 1 (i) It was Raney who first proved that a complete lattice is supercontinuous if and only if it is a completely distributive lattice (of course Raney didn't use the term 'supercontinuous lattice'. Banaschewski first used this term). However the equivalence of complete distributivity and supercontinuity heavily depends on the Axiom of Choice. In fact, the definition of completely distributive lattices itself involves the use of function of choice. Thus if we want to do constructive work we should adopt supercontinuous lattices as a replacement of completely distributive lattices. Using an equivalent condition one can define supercontinuous lattice in a topos (see [5] for more details about constructive complete distributivity). Fortunately in most of the cases we need the supercontinuity instead of complete distributivity.

(ii) By the definition of $\lhd$ it follows that $\{x \in L \mid x \lhd 0_L\}$ is empty, where 0_L is the bottom element of L.

A complete lattice L is called totalcontinuous if $a = \vee\{x \in L \mid x \lhd x \leq a\}$ holds for every $a \in L$, in other words, if the supercompact elements are join-dense in L. Obviously every totalcontinuous lattice is supercontinuous.

The relation $\lhd$ on L is said to be stable if $a \lhd b$ and $a \lhd c$ imply $a \lhd b \wedge c$.

A stably supercontinuous lattice is a supercontinuous lattice with the two properties (1) The top element of L is supercompact, i.e., $1_L \lhd 1_L$ and (2) $\lhd$ is stable. A stably totalcontinuous lattice is a totalcontinuous which satisfies the above two conditions (1) and (2).

Remark 2 (i) For each supercontinuous lattice L the relation $\lhd$ satisfies the interpolation property, i.e., if $a \lhd b$ then there exists $c \in L$ such that $a \lhd c \lhd b$ (see [6])

(ii) If S is a meet-semilattice with a top element 1s, then the poset DS of all lower sets of S is a stably totalcontinuous lattice. Conversely, for every stably totalcontinuous lattice L there is a meet-semilattice S (e.g., the set of all supercompact elements of L) such that $L \cong DS$.

(iii) Every supercontinuous lattice is a frame. The proof of this indication is free of Axiom of Choice. By using Zorn's lemma one can show that every supercontinuous lattice is a spatial frame, i.e. every element is a meet of prime elements.

(iv) For each element a of a complete lattice L, we write $\beta(a) = \{x \mid x \lhd a\}$. Thus $\beta : L \to DL$ is a function from L to the set of all lower sets of L.

Proposition 2.3 *A complete lattice is a stably supercontinuous lattice if and only if it is a retract of some stably totalcontinuous lattice by maps which preserve arbitrary joins and finite meets.*

In [7] Banaschewski proved that the open set lattices of injective T_0-spaces are exactly the stably supercontinuous lattices. By Scott's result a topological space X is an injective T_0-space if and only if there is a continuous lattice L such that X is homeomorphic to $(L, \sigma(L))$ where $\sigma(L)$ is the Scott topology on L, so X must be sober (see [6]). Thus we have the following lemma.

Lemma 2.4 *A topological space X is an injective T_0-space if and only if it is sober and its open set lattice $O(X)$ is a stably supercontinuous lattice.*

3.The Eilenberg-Moore category $\mathbf{Top}^T$

Let Slat be the category of all meet-semilattices which have top elements and maps preserving finite meets and top elements. Slat is obviously a concrete category satisfying conditions $(*)$ and $(**)$ of section 2. Thus by lemma 2.1 there is a monad $T = (T, \eta, \mu)$ on Top. T is the composition functor $T = \mathrm{Spec} \circ O : \mathrm{Top} \to \mathrm{Top}, \eta : id \to T$ is the natural transformation which assigns to each space X the map $\eta_X : X \to T(X)$ such that for any $x \in X, \eta_X(x) : O(X) \to 2$ with $(\eta_X(x))(U) = 1$ iff $x \in U$ for each $U \in O(X)$. $\mu : T^2 \to T$ is the natural transformation which assigns to each space X the map $\mu_X : T^2X \to TX$ such that for any $f \in T^2X, \mu_X(f) : O(X) \to 2$ is defined by $\mu_X(f)(U) = 1$ iff $f(\sigma(U)) = 1$ (recall that $\sigma(U)$ is an open set of TX).

An element $f \in \mathrm{Spec}(S)$ is thus a meet-semilattice homomorphism from S to 2 which sends the top element of S to 1. We call the elements of Spec(S) characters of S.

Lemma 3.1 *Let S be a meet-semilattice with a top elements. Then the space Spec(S) satisfies the following two conditions: (1) Spec(S) is sober; (2) O(Spec(S)) is a stably totalcontinuous lattice.*

Proof First notice that for the case of meet-semilattice,$\{\sigma(x) \mid x \in S\}$ is a basis of the topology of Spec(S) because $\sigma(x) \bigcap \sigma(y) = \sigma(x \wedge y)$. For each $a \in S$ the character f_a defined by $f_a^{-1}(1) = \uparrow a = \{x \in S \mid x \geq a\}$ is in $\sigma(a)$. Now if $B \subseteq S$ such that $\cup_{x \in B} \sigma(x) \supseteq \sigma(a)$, then there is a $x \in B$ such that $f_a \in \sigma(x)$.So $a \leq x$, and hence $\sigma(a) \subseteq \sigma(x)$. Hence $\sigma(a)$ is supercompact because all $\sigma(x)$ constitute a basis of $O(\mathrm{Spec}(S))$. In particular $\sigma(1_S) = \mathrm{Spec}(S)$ is supercompact. As $\{\sigma(x) \mid x \in S\}$ is a basis of $O(\mathrm{Spec}(S))$ and $\sigma(x) \cap \sigma(y) = \sigma(x \wedge y)$ it follows immediately that $O(\mathrm{Spec}(S))$ is a stably totalcontinuous lattice. So the condition (2) is satisfied.

Spec(S) is clearly T_0. To see the soberness, let B be a non-empty irreducible closed set of Spec(S). There is a set $A \subseteq S$ such that $B = \cap\{\sigma^c(x) \mid x \in A\}$, where $\sigma^c(x)$=Spec(S) $\setminus \sigma(x)$. As $x \leq y$ implies $\sigma^c(x) \supseteq \sigma^c(y)$ we can assume that A is a lower set of S. Hence $B = \{f \in \text{Spec}(S) \mid A \subseteq f^{-1}(\{0\})\}$. Define a map $f_A : S \to 2$ which satisfies $f_A^{-1}(\{0\}) = A$. Obviously $A \neq S$. Suppose $x \wedge y \in A$ then $B \subseteq \sigma^c(x \wedge y) = (\sigma(x) \cap \sigma(y))^c = \sigma^c(x) \cup \sigma^c(y)$. Hence either $B \subseteq \sigma^c(y)$, or $B \subseteq \sigma^c(y)$, which then deduce that either $x \in A$ or $y \in A$. From this it follows that f_A is a character of S and obviously $f_A \in B$. Moreover from the equation $B = \{f \in \text{Spec}(S) \mid A \subseteq f^{-1}(\{0\})\}$ it follows that f_A is a generic point of B, i.e., cl($\{f_A\}$) $= B$. So Spec(S) is sober.

Lemma 3.2 *If X is a sober space such that $O(X)$ is a stably totalcontinuous lattice, then there is a meet-semilattice S which has a top element such that $X \cong$ Spec(S).*

Proof Let X be a topological space satisfying the above conditions. Put $S = \{U \in O(X) \mid U \lhd U\}$. Then S is a sub-meet-semilattice of $O(X)$ and contains the top element X. There is an natural function $\lambda : X \to \text{Spec}(S)$ which sends $a \in X$ to λ_a such that $\lambda_a(U) = 1$ iff $a \in U (U \in S)$. Now if $f \in \text{Spec}(S)$,let $W = \cup\{V \in S \mid f(V) = 0\}$ (W is not necessarily in S). We show that W is a prime open set of X. In fact suppose that U and U' are open sets such that $U \cap U' \subseteq W$ and $U \not\subseteq W, U' \not\subseteq W$, then there exist $V \in S, V \subseteq U, V \not\subseteq W$ and $V' \in S, V' \subseteq U', V' \not\subseteq W$. So $V \cap V' \subseteq U \cap U' \subseteq W$. Since $V \cap V' \in S$, there is a $B \in S, f(B) = 0$ and $V \cap V' \subseteq B$. Thus $f(V \cap V') = 0$. However $V \not\subseteq W$ and $V' \not\subseteq W$ imply that $f(V) = 1$ and $f(V') = 1$, so $f(V \cap V') = 1$. This contradiction proves that W is a prime open set. As X is sober, W_f^c has a unique generic point, denoted by ξ_f . Thus we have a function $\xi : \text{Spec}(S) \to X$, where $\xi(f) = \xi_f$. For each $x \in X$,from that S is a basis of $O(X)$ it follows that $\xi(\lambda(x)) = x$. So $\xi \circ \lambda = id_X$. For each $f \in \text{Spec}(S)$, if $f(V) = 1$ for a $V \in S$, then $V \not\subseteq \cup\{U \in S \mid f(U) = 0\}$ because V is supercompact. Hence $\xi_f \in V$, so $\lambda(\xi(f))(V) = 1$. Conversely, if $\lambda(\xi(f))(V) = 1$, then $\xi(f) \in V$, so $V \not\subseteq \cup\{U \in S \mid f(U) = 0\}$, thus $f(V) = 1$. This shows that $\lambda \circ \xi = id_{Spec(S)}$. So λ and ξ are one-to-one maps. In addition, for each $V \in S$, $\xi(\sigma(V)) = V, \lambda(V) = \sigma(V)$. Thus ξ and λ are both open maps. Hence ξ sets up a homeomorphism between X and Spec(S).

Proposition 3.3 *A topological space X is a sober space and $O(X)$ is a stably totalcontinuous lattice if and only if $X \cong$ Spec(S) for some meet-semilattice S with a top element.*

Corollary 3.4 *For each meet-semilattice S which has a top element, the spectral space Spec(S) is an injective T_0-space.*

Recall that an algebra for T is a pair (X, h) with X a topological space and $h : TX \to X$ a continuous map, such that $h \circ \eta_X = id_X$ and $h \circ Th = h \circ \mu_X$, where $\eta_X : X \to TX$ and $\mu_X : T^2X \to TX$.

Lemma 3.5 *Let X be an injective T_0 space, then for character $f : O(X) \to 2$ of $O(X)$, $\cup\{U \mid f(U) = 0\}$ is a prime open set.*

Proof let $W_f = \{U \in O(X) \mid f(U) = 0\}$. Suppose that V and E are two open sets such that $V \cap E \subseteq W_f$ and $V \not\subseteq W_f, E \not\subseteq W_f$. By lemma 2.4, $O(X)$ is a stably supercontinuous lattice, so there are $V' \in O(X)$ and $E' \in O(X)$, such that $V' \lhd V, E' \lhd E$ and $V' \not\subseteq W_f, E' \not\subseteq W_f$. So $f(V') = 1, f(E') = 1$. Since f preserves finite meets, $f(V' \cap E') = 1$. On the other hand, the relation $\lhd$ in $O(X)$ is stable, so $V' \cap E' \lhd V \cap E$. From $V \cap E \subseteq W_f$ it follows that there exists a $U \in O(X)$ such that f(U)=0 and $V' \cap E' \subseteq U$. But this implies that $f(V' \cap E') = 0$ which contradicts that $f(V' \cap E') = 1$. Hence W_f is prime.

Now by the above lemma, if X is an injective T_0 space, there is a map $m : \mathrm{Spec}(O(X)) \to X$, where for each $f \in \mathrm{Spec}(O(X)), m(f)$ is the unique generic point of the irreducible closed set W_f^c. Obviously $m(f) \in U \in O(X)$ implies $f(U) = 1$.

Lemma 3.6 *If (X, h) is a T-algebra, then X is an injective T_0-space and $h = m$.*

Proof If (X, h) is a T-algebra, then X is a retract of TX, which is an injective T_0-space, so X is an injective T_0-space. Now let $f \in TX = \mathrm{Spec}(O(X))$ be any character of $O(X)$. Suppose $h(f) \in U \in O(X)$, then as h is continuous there is an open set V of X such that $f \in \sigma(V) \subseteq h^{-1}(U)$. Then $f(V) = 1$. If $x \in V$, then $\eta_X(x) \in \sigma(V)$, so $h \circ \eta_X(x) = x \in h(\sigma(V)) \subseteq U$. Hence $V \subseteq U$, and from $f(U) \geq f(V) = 1$ we see that $f(U) = 1$. It follows that $h(f) \in W_f^c$ where $W_f = \cup\{E \in O(X) \mid f(E) = 0\}$. On the other hand, by the definition of $m(f)$ we see that $\eta_X(m(f)) \leq f$, this implies that $\eta_X(m(f)) \in cl(\{f\})$ holds in the space TX. Thus $m(f) = h(\eta_X(m(f))) \in h(\mathrm{cl}(\{f\})) \subseteq \mathrm{cl}(\{h(f)\})$. Since $m(f)$ is the unique generic point of W_f^c and $h(f) \in W_f^c$, so $h(f) = m(f)$.

Lemma 3.7 *For an injective T_0 space X, the map $m : \mathrm{Spec}(O(X)) \to X$ defined above is continuous.*

Proof Suppose $f \in \mathrm{Spec}(O(X))$ and $m(f) \in U \in O(X)$. Then, as $O(X)$ is supercontinuous, there is a $V \in O(X)$ with $m(f) \in V \lhd U$. So $f(V) = 1$, i.e., $f \in \sigma(V)$. Now for each $g \in \sigma(V), m(g) \in U$, otherwise the relations $U \subseteq W_g = \cup\{E \in O(X) \mid g(E) = 0\}$, together with $V \lhd U$ would imply that $V \subseteq E$ for some $E \in O(X)$ with $g(E) = 0$, which further implies $g(V) = 0$, but this contradicts to that $g \in \sigma(V)$ which means $g(V) = 1$. Thus f has a neighbourhood $\sigma(V)$ contained in $m^{-1}(U)$. So m is continuous.

Lemma 3.8 *For any injective T_0 space X, the pair (X, m) is a T-algebra.*

Proof By lemma 3.7 m is a continuous map. Also it is clear that $m \circ \eta_X = id_X$. Thus we only need to prove the equation $m \circ Tm = m \circ \mu_X$. Let $f \in \mathrm{Spec}(O(\mathrm{Spec}(O(X))))$, then $m(Tm(f))$ is the unique generic point of $W^c_{Tm(f)}$ and $m(\mu_X(f))$ is the unique generic point of $W^c_{\mu_X(f)}$, where $W_{Tm(f)} = \cup\{U \in O(X) \mid Tm(f)(U) = 0\}$ and $W_{\mu_X(f)} = \cup\{U \in O(X) \mid \mu_X(f)(U) = 0\}$. If we can show $W_{Tm(f)} = W_{\mu_X(f)}$ then $m(Tm(f)) = m(\mu_X(f))$. Let $U \in O(X)$ with $\mu_X(f)(U) = f(\sigma(U)) = 0$. As $m^{-1}(U) \subseteq \sigma(U)$ always holds, so $0 = f(m^{-1}(U)) = Tm(f)(U)$ (note that $Tm(f) = f \circ m^{-1}$). Thus $W_{\mu_X(f)} \subseteq W_{Tm(f)}$. Conversely suppose $U \in O(X)$ such that $Tm(f)(U) = f(m^{-1}(U)) = 0$. For any $V \in O(X)$ with $V \lhd V$, we have $\sigma(V) \subseteq m^{-1}(U)$. In fact if $m(g) \notin U$, then $U \subseteq W_g$, so $V \subseteq E$ for some $E \in O(X)$ with $g(E) = 0$. This then induces $g(V) = 0$ which means $g \notin \sigma(V)$. Now $f(\sigma(V)) \leq f(m^{-1}(U)) = 0$ implies $\mu_X(f)(V) = f(\sigma(V)) = 0$,i.e.$V \subseteq W_{\mu_X(f)}$. Since $O(X)$ is supercontinuous,$U = \cup\{V \in O(X) \mid V \lhd U\} \subseteq W_{\mu_X}(f)$. Hence $W_{Tm(f)} \subseteq W_{\mu_X(f)}$. Thus we proved that $W_{Tm(f)} = W_{\mu_X(f)}$.

Combining the above conclusions we get the following result.

Theorem 3.9 *A pair (X, h) is a T-algebra if and only if X is an injective T_0-space and $h = m$.*

Recall that the Eilenberg-Moore category Top^T is the category whose objects are T-algebras, and whose morphisms are T-algebra morphisms, where a T-algebra morphism $f : (X, h) \to (Y, k)$ is a continuous map $f : X \to Y$ such that $k \circ Tf = f \circ h$.

Definition 3.10 *A continuous map $\gamma : X \to Y$ from topological space X to Y is called strongly algebraic if the map $\gamma^{-1} : O(Y) \to O(X)$ preserves the relation $\lhd$.*

This terminology is justified by the fact that every strongly algebraic map is an algebraic map in the sense of [4].

Lemma 3.11 *If X and Y are two injective T_0-spaces, then a continuous map $\gamma : X \to Y$ is a T-algebra morphism from (X, m) to (Y, m) if and only if it is strongly algebraic.*

Proof Suppose that γ is strongly algebraic. We want to show that $\gamma \circ m = m \circ T\gamma$. Let $f \in TX = \text{Spec}(O(X))$. Then $T\gamma(f) = f \circ \gamma^{-1} : O(Y) \to 2$. For any $U \in O(Y)$, if $\gamma(m(f)) \in U$ then $m(f) \in \gamma^{-1}(U)$, so $f(\gamma^{-1}(U)) = 1$. But $T\gamma(f)(U) = f(\gamma^{-1}(U))$, so $\gamma(m(f)) \in W^c_{T\gamma(f)}$,which implies that $\gamma(m(f)) \in cl(\{T\gamma\})$. Now if we can show that $m(T\gamma(f)) \in cl(\{\gamma(m(f))\})$, then $m(T\gamma) = \gamma(m(f))$. Suppose $m(T\gamma(f)) \in U \in O(Y)$. If $\gamma(m(f)) \notin U$, then $\gamma^{-1}(U) \subseteq W_f$. Now, by the assumption, for any $V \lhd U, V \in O(Y), \gamma^{-1}(V) \lhd \gamma^{-1}(U)$, so there exists $E \in O(X)$ with $f(E) = 0$ and $\gamma^{-1}(V) \subseteq E$, hence $T\gamma(f)(V) = f(\gamma^{-1}(V)) = 0$ which indicates that $V \subseteq W_{T\gamma(f)}$. So $U \subseteq W_{T\gamma(f)}$, which contradicts to that $m(T\gamma(f)) \in U$. Hence $m(T\gamma(f)) \in cl(\{\gamma(m(f))\})$.

Conversely suppose γ is a strongly algebraic map, $V \lhd U$ holds in $O(Y)$, and $\cup\{E_i \mid i \in I\}$ is an open cover of $\gamma^{-1}(U)$. From that γ is a T-algebraic map it easily follows that for any $f \in TX$ the relation $T\gamma(f)(V) = 1$ implies $m(f) \in \gamma^{-1}(U)$. Define $f_V \in TX$ by $f_V(E) = 1$ iff $E \supseteq \gamma^{-1}(V)$. As $T\gamma(f_V)(V) = f_V(\gamma^{-1}(V)) = 1$, so $m(f_V) \in \gamma^{-1}(V) \subseteq \gamma^{-1}(U) \subseteq \cup\{E_i \mid i \in I\}$. So there exists E_i with $m(f) \in E_i$, which then implies that $\gamma^{-1}(V) \subseteq E_i$. Hence $\gamma^{-1}(V) \lhd \gamma^{-1}(U)$. Therefore γ is a strongly algebraic map.

Theorem 3.12 *The Eilenberg-Moore category Top^T is the Inj_0 of all injective T_0-spaces and strongly algebraic maps between them.*

References:

[1] HOFFMANN R E. *Monads induced by topological spaces and their Eilenberg-Moore categories* [J]. Seminarberich Math. der Fern Universit' at Hagen, 1984, **19**(1): 207–216.

[2] JOHNSTONE P T. *Stone Spaces* [M]. Cambridge University Press, Cambridge, 1982.

[3] SOBREL M. *CABool is monadic over almost all categories* [J]. J.Pure and Applied Alg., 1992, **77**: 207–218.

[4] SIMMONS H. *A couple of triples* [J]. Top. And Appl., 1982, **13**: 201–223.

[5] FAWCETT B and WOOD R J. *Constructive complete distributivity* [J]. I. Math. Proc. Camb. Phil. Soc., 1990, **107**: 81–89.

[6] GIERZ G. etal. *A Compendium of Continuous Lattices* [M]. Springer-Verlag, Berlin, 1980.

[7] BANASCHEWSKI B. *On the topologies of injective spaces, in: Continuous Lattices and Their Applications* [M]. Marcel Dekker, New York, 1985, 1–8.

Inj_0 在 Top 上的 Monadicity

赵 东 升[1], 赵 彬[2]

(1. 南洋理工大学数学系，新加坡； 2. 陕西师范大学数学系，西安 710062)

摘 要: 本文主要证明了全体内射 T_0- 空间及强代数映射构成的范畴 Inj_0 恰是 Eilenberg-Moore 范畴 Top^T, 这里 T 是 Top 与 Slat 之间的一对偶伴随导出的 monad, 由此推出 Inj_0 在 Top 上是 monadic 的.