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The Monotone Iterative Roots of a Class of Self-Mappings on the Interval *

LIU Xin-he[1,2]

(1. Dept. of Math., Guangxi Univ., Naning 530004, China;

2. Inst. of Math., Shantou Univ., Guangdong 515063, China)

Abstract: Let $I = [0,1]$, $0 < a < b < 1$. Let $\Phi_{ab} \equiv \{F \in C^0(I)$: both $F|[0,a]$ and $F|[b,1]$ are strictly increasing, and $F|[a,b]$ is constant $\}$. In this paper we discuss necessary and sufficient conditions for $F \in \Phi_{ab}$ to have monotone iterative roots.

Key words: continuous self-map; fixed point; iterative root.

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1. Introduction

Let X be a topological space and $F \in C^0(X)$. For some integer $n \geq 2$, F is said to have an n-order iterative root if there exits $f \in C^0(X)$ such that $f^n = F$, where f^n denotes the n-th iterate of f.

The problems of iterative roots play an important role in the study of iterative functional equations, i.e. the equations containing iterates of an unknown function. However, the researches on the iterative roots of the continuous self-maps on the interval usually limit to the strictly monotone cases or the strictly piecewise monotone cases, rarely touching upon the other cases[6]. In this paper we discuss the monotone iterative roots of the monotone increasing continuous self-maps with a level segment on the interval.

Let $I = [0,1]$, and a, b be given real numbers, where $0 < a < b < 1$. Let $\Phi_{ab} \equiv \{F \in C^0(I)$: both $F|[0,a]$ and $F|[b,1]$ are strictly increasing, and $F|[a,b]$ is constant $\}$. Let $\mathcal{F}_{ab} = \{F \in \Phi_{ab} : \mathrm{Fix}(F) \cap [(0,a) \cup (b,1)] = \emptyset\}$. We denote $E = Fix(F) \cup \{0,1\}$, $\bar{a} = \max(E \cap [0,a))$, and $\bar{b} = \min(E \cap (b,1])$.

Now we state the main results of this paper.

Theorem 1 *Let $F \in \Phi_{ab}$. Then for each integer $n \geq 2$, F has a monotone increasing*

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Biography: LIU Xin-he (1969-), male, born in Xiangtan city, Hunan province. Ph.D., currently a lecturer of Guangxi University.

n-order iterative root if and only if one of the following three conditions holds: (1) $F(a) \in [a,b]$; (2) $F(\bar{a}) > b$; (3) $F(\bar{b}) < a$.

Theorem 2 *Let $F \in \mathcal{F}_{ab}$. Then for each integer $n \geq 2$, F has a monotone decreasing n-order iterative root if and only if n is even, $F(a) \in (a,b)$, and one of the following conditions holds:*

(1) $F(0) = 0$, $F(1) = 1$;

(2) $F(0) > 0$, $F(1) < 1$, and one of the following four conditions holds: (2.1) $m_0 = m_1$; (2.2) $m_0 = m_1 + 1$ and $F^{m_0}(0) > a$; (2.3) $m_0 = m_1 - 1$ and $F^{m_1}(1) < b$; and (2.4) $|m_0 - m_1| = 1$, $F^{m_0}(0) = a$, $F^{m_1}(1) = b$, and $n = 2$, where $m_i = min\{m : m > 0, F^m(i) \in [a,b]\}$, $(i = 0,1)$.

Theorem 3 *Let $F \in \Phi_{ab} \backslash \mathcal{F}_{ab}$. Suppose there exists a $y_0 \in \mathrm{Fix}(F|(a,b))$. Let $E_1 = [0, y_0) \cap E$ and $E_2 = (y_0, 1] \cap E$. Then for each given even integer $n \geq 2$, F has a monotone decreasing n-order iterative root if and only if*

(1) There exits an order-reserving one to one map $D : E_1 \to E_2$, and

(2) For any two consecutive points e_1 and e_2 in E_1 with $e_1 < e_2$, $F(x) - x$ is positive (or negative) on $(D(e_2), D(e_1))$ when it is negative (or positive) on (e_1, e_2), and either $(F(0) - 0)(F(1)) - 1) < 0$ or $F(0) + 1 - F(0) = 0$.

1. Necessary and sufficient conditions for $F \in \Phi_{ab}$ having n-order increasing iterative roots

Lemma 1.1 *Suppose $F \in \Phi_{ab}$ has an n-order iterative root f, then both $f^i|[0,a]$ and $f^i|[b,1]$ are strictly increasing, $i = 1, 2, \cdots, n$.*

Proof It is easy to see. □

Lemma 1.2 *Suppose $F \in \mathcal{F}_{ab}$, and f is an n-order iterative root of F. If there exists some $x_0 \in [a,b]$ such that $f(x_0) \notin [a,b]$, then $f|[a,b]$ is a constant function and $f(I) \cap (a,b) = \emptyset$.*

Proof (1) If $f(x_0) > b$, then $f|[a,b]$ is a constant function, otherwise there exists a point $y_0 \in (a,b)$ such that $f(y_0) \neq f(x_0)$ and $f(y_0) > b$. By Lemma 1.1 we have $F(y_0) \neq F(x_0)$, which is a contradiction.

Now we claim $f(I) \subset [b,1]$. In fact, if there exist one point $z_0 \in I$ such that $f(z_0) < b$, then there exist two points $x_1, x_2 \in [0,a]$ or $x_1, x_2 \in [b,1]$ such that $f(x_1) \neq f(x_2)$ and $f(x_1), f(x_2) \in [a,b]$. So $F(x_1) = F(x_2)$, which is a contradition. Thus we have $f(I) \cap (a,b) = \emptyset$.

(2) By a similar argument, it follows that $f|[a,b]$ is a constant function and $f(I) \cap (a,b) = \emptyset$ if $f(x_0) < a$. □

Lemma 1.3 *Suppose $F \in \mathcal{F}_{ab}$ has an increasing iterative root. Then one of the following three conditions holds: (1) $F(a) \in [a,b]$; (2) $F(0) > b$; (3) $F(1) < a$.*

Proof Suppose f is an n-order increasing iterative root of F. It suffices to show that either (2) or (3) holds if (1) doesn't holds. Assume $F(a) \notin [a,b]$ then both $F|[a,b]$ and

$f|[a,b]$ have no fixed point. By Lemmas 1.1 and 1.2, we must have either $f(I)\subset[b,1]$ or $f(I)\subset[0,a]$. If $f(I)\subset[b,1]$ then $f^{n-1}(0)\geq b$. It follows from Lemmas 1.1 and 1.2 that $F(0)=f(f^{n-1}(0))\geq f(b)>f(0)\geq b$. By a similar argument, it follows that $F(1)<a$ if $f(I)\subset[0,a]$. □

Lemma 1.4 *Suppose $F\in\mathcal{F}_{ab}$. If either $F(0)>b$ or $F(1)<a$, then for each given integer $n\geq 2$, F has an n-order increasing iterative root.*

Proof It suffices to construct an n-order increasing iterative root of F. Without loss of generality, we assume $F(0)>b$.

We choose arbitrarily n points $b_1,b_2,\cdots,b_{n-1},b_n\in[b,F(0)]$ with $b\leq b_1<b_2<b_{n-1}<b_n=F(0)$. Let $b_j=F(b_{j-n})$ for $j\geq n+1$.

(1) For $i=1,2,\cdots,n-1$, let $f_i:[b_i,b_{i+1}]\rightarrow[b_{i+1},b_{i+2}]$ be a strictly increasing continuous function with $f_i(b_i)=b_{i+1}$ and $f_i(b_{i+1})=b_{i+2}$.

(2) For $i\geq n$ we define successively $f_i:[b_i,b_{i+1}]\rightarrow[b_{i+1},b_{i+2}]$ by $f_i=F\circ f^{-1}_{i-n+1}\circ\cdots\circ f^{-1}_{i-2}\circ f^{-1}_{i-1}$.

(3) We define $f:I\rightarrow I$ by $f(1)=1$ and $f(x)=f^{-1}_{i-n+1}\circ\cdots\circ f^{-1}_{i-2}\circ f^{-1}_{i-1}\circ F(x)$ if $F(x)\in[b_i,b_{i+1})$ for some $i\geq n$. It is easy to verify that f is an n-order increasing iterative root of F. □

Lemma 1.5 *Suppose $F\in\mathcal{F}_{ab}$. If $F(a)=r\in[a,b]$, then for each given integer $n\geq 2$, F has an n-order increasing iterative root.*

Proof It is obvious that r is the unique fixed point of $F|[a,b]$. The proof will be carried out in a number of stages:

(1.1) Assume $r\in(a,b)$ and $F(1)=1$. Choose arbitrarily $n-1$ points $t_1,t_2,\cdots,t_{n-1}\in(r,b)$ satisfying $t_1<t_2<\cdots<t_{n-2}<t_{n-1}$. Let $t_n=b$ and $t_0=r$. For $j>n$ we define successively $t_j=F^{-1}(t_{j-n})$.

For $i=1,\cdots,n-1$, let $f_i:[t_i,t_{i+1}]\rightarrow[t_{i-1},t_i]$ be a strictly increasing continuous function with $f_i(t_i)=t_{i-1}, f_i(t_{i+1})=t_i$. Let $f_0:[t_0,1]\rightarrow[t_0,1]$ be the constant function, where the constant is r. For $i\geq n$, we define successively $f_i:[t_i,t_{i+1}]\rightarrow[t_{i-1},t_i]$ by $f_i=f^{-1}_{i-1}\circ f^{-1}_{i-2}\circ\cdots\circ f^{-1}_{i-n+2}\circ f^{-1}_{i-n+1}\circ F$.

Define $f_{r1}:[r,1]\rightarrow[r,1]$ by $f_{r1}(x)=\begin{cases}1 & \text{if } x=1;\\ f_i(x) & \text{if } x\in[t_i,t_{i+1}),\ i=0,1,\cdots\end{cases}$

It is easy to verify that f_{r1} is an n-order increasing iterative root of $F|[r,1]$.

(1.2) Assume $r\in(a,b)$ and $F(1)<1$. Choose arbitrily a real number $s>1$, Let $F_s:[r,s]\rightarrow[r,s]$ be an increasing continuous function satisfying that $F_s|[1,s]$ is a strictly increasing continuous function, $F_s|[r,1]=F|[r,1], F_s(s)=s$, and $Fix(F_s)\cap(1,s)=\emptyset$. It follows from (1.1) that F_s has an n-order increasing iterative root $f_{rs}:[r,s]\rightarrow[r,s]$ with $f_{rs}(r)=r$, $f_{rs}(s)=s$ and $f_{rs}(x)<x$ if $x\in(r,s)$. Let $f_{r1}=f_{rs}|[r,1]$ then f_{r1} is an n-order increasing iterative root of $F|[r,1]$.

(1.3) In the same way, we can prove that $F|[0,r]$ has an n-order increasing iterative root $f_{0r}:[0,r]\rightarrow[0,r]$ satisfying $f_{0r}(r)=r$ if $r\in(a,b)$.

Define $f:I\rightarrow I$ by $f|[0,r]=f_{0r}$ and $f|[r,1]=f_{r1}$, then f is an n-order increasing iterative root of F.

(2) Assume either $r = a$ or $r = b$. By the results as above and Hardy-Böedwadt theorem (see [2]), it follows that there exists an n-order increasing iterative root of F.

Remark 1.1 According to the proof of Lemma 1.5, $F \in \mathcal{F}_{ab}$ with $F(a) = r \in [a, b]$ has an n-order increasing iterative root $f : I \to I$ which has the following properties:(1) there exists some subinterval $[u, v] \subset [a, b]$ containing r such that $f([u, v]) = \{r\}$, both $f|[0, u]$ and $f|[v, 1]$ are strictly increasing, and $((0, u) \cup (v, 1)) \cap Fix(f) = \emptyset$; (2) $u = r$ if $r = a$, $r > u$ if $r > a$, $v = r$ if $r = b$, and $r < v$ if $r < b$; (3) $f(0) = 0$ if $F(0) = 0$, $f(0) > 0$ if $F(0) > 0$, $f(1) = 1$ if $F(1) = 1$, and $f(1) < 1$ if $F(1) < 1$.

Proof of Theorem 1 By Lemmas 1.3–1.5, for each integer $n \geq 2$, $F|[\bar{a}, \bar{b}]$ has a monotone increasing n-order iterative root if and only if one of the following three conditions holds: (1) $F(a) \in [a, b]$; (2) $F(\bar{a}) > b$; (3) $F(\bar{b}) < a$.

Assume $x < x^*$ are two consecutive points in E. If f is a monotone increasing n-order iterative root of F, then $(f|[x, x^*])^n = F|[x, x^*]$, and the converse also holds. In fact, it suffices to verify $Fix(F) = Fix(f)$. For any $y_0 \in I$, if $f(y_0) > y_0$ then $F(y_0) = f^n(y_0) \geq f^{n-1}(y_o) \geq \cdots \geq f(y_0) > y_0$ since f is a monotone increasing function; by a similar argument if $f(y_0) < y_0$ then $F(y_0) < y_0$. Thus $Fix(F) \subset Fix(f)$. Hence $Fix(F) = Fix(f)$. By Hardy-Böedwadt theorem ([2]), F has a monotone increasing n-order iterative root if and only if $F|[\bar{a}, \bar{b}]$ has a monotone increasing n-order iterative root. This completes the proof. □

2. Necessary and sufficient conditions for $F \in \mathcal{F}_{ab}$ having n-order decreasing iterative roots

Lemma 2.1 *Suppose $F \in \mathcal{F}_{ab}$. If $F(a) \notin (a, b)$ then F has no decreasing iterative root.*

Proof Assume, on the contrary, that f is an n-order decreasing iterative root of F.

(1) If $F(a) \notin [a, b]$, then by Lemma 1.2 we must have either $f(I) \subset [b, 1]$ or $f(I) \subset [0, a]$. If $f(I) \subset [b, 1]$, then by Lemma 1.1 and $Fix(F|(b, 1)) = \emptyset$, $f|[b, 1]$ is strictly increasing, which is a contradition. By a similar argument, it leads to a contradition if $f(I) \subset [0, a]$.

(2) If $F(a) = b$, then by Lemma 1.1 and $Fix(F) \cap (0, 1) = \{b\}$, $Fix(f) = \{b\}$, thus for each $x \in [a, b], f(x) \geq b$. On the other hand, by Lemma 1.2 we have $f([a, b]) \subset [a, b]$, so for each $x \in [a, b], f(x) \leq b$. Therefore $f([a, b]) = \{b\}$. Choose arbitrarily two points x_1, x_2 with $x_1 \neq x_2$ in $(b, 1)$ such that $f(x_1), f(x_2) \in (a, b)$, then $f^2(x_1) = f^2(x_2)$, it follows that $F(x_1) = F(x_2)$. Since $F|[b, 1]$ is strictly increasing, it is a contradition.

(3) By a similar argument, it leads to a contradition if $F(a) = a$.

Thus F has no decreasing iterative root. □

Lemma 2.2 *Suppose $F \in \mathcal{F}_{ab}, F(a) = r \in (a, b), F(0) > 0$ and $F(1) < 1$. Let $m_i = min\{m : m > 0, F^m(i) \in [a, b]\}, (i = 0, 1)$. If F has an n-order decreasing iterative root f, then (1^0) n is even, $|m_0 - m_1| \leq 1$; (2^0) if $m_0 - m_1 = 1$ then $F^{m_0}(0) = a$ implies $F^{m_1}(1) = b$ and $n = 2$; (3^0) if $m_0 - m_1 = -1$ then $F^{m_1}(1) = b$ implies $F^{m_0}(0) = a$ and $n = 2$.*

Proof (1) It is obvious that n is even and r is the unique fixed point of f.

It follows from Lemma 1.2 that $f([a,b]) \subset [a,b]$. Since F is a monotone increasing function and $f(0) \leq 1$, $F^{m_1}(f(0)) \leq F^{m_1}(1)$. It follows that $f^{nm_1+1}(0) \in [r,b]$, therefore $f^{nm_1+n}(0) \in f^{n-1}([r,b]) \subset [a,b]$, thus $m_0 \leq m_1 + 1$. By a similar argument, we have $m_1 \leq m_0 + 1$. Hence $|m_0 - m_1| \leq 1$.

(2) Now we consider the case $m_0 - m_1 = 1$ and $F^{m_0}(0) = a$. we will show that $n = 2$ and $F^{m_1}(1) = b$. Since $f^n([a,b]) = F([a,b]) = \{r\}$, $f([a,b]) \neq [a,b]$, therefore $|f(a) - b| + |f(b) - a| \neq 0$. We claim that $f^{nm_0-1}(0) \geq b$. In fact, if that $f^{nm_0-1}(0) < b$, then $a = F^{m_0}(0) \geq f(b) \geq a$, therefore $f(x) = a$ for each $x \in [f^{nm_0-1}(0), b]$. On the other hand, since $m_0 \geq 2$, we have $f(0) > b$, and hence there exist two distinct points $x_1, x_2 \in [0,a]$ such that $f(x_1), f(x_2) \in [f^{nm_0-1}(0), b]$. It follows that $F(x_1) = F(x_2)$, which is a contradiction.

Assume, on the contrary, that $n \geq 4$. Since $f(0) \leq 1$, we have $f^{nm_0-2}(1) \geq f^{nm_0-1}(0) \geq b$, thus $f^{nm_0-2}(1) < f^{nm_0-4}(1) < \cdots < f^{nm_0-n}(1)$. It follows that $F^{m_1}(1) > b$, which is a contradiction. Hence $n = 2$.

It follows from the claim as above that $f^{2m_0-1}(0) \geq b$, thus $F^{m_1}(1) = f^{2m_0-2}(1) \geq f^{2m_0-2}(f(0)) = f^{2m_0-1}(0) \geq b$, on the other hand $F^{m_1}(1) \leq b$, Hence $F^{m_1}(1) = b$.

(3) By a similar argument, (3^0) holds. □

Lemma 2.3 *Suppose $F \in \mathcal{F}_{ab}$. If $F(a) = r \in (a,b)$, $F(0) = 0$ and $F(1) = 1$, then for each given even $n \geq 2$, F has an n-order decreasing iterative root.*

Proof According to Lemma 1.5 and Remark 1.1, it suffices to show that F has a 2-order decreasing iterative root. It is obvious that $r \in Fix(F)$.

We choose arbitrarily two points $d \in (a,r)$ and $c \in (r,b)$. For $k = 0, 1, \cdots$, let $x_{2k} = F^{-k}(d)$; $x_{2k+1} = F^{-k}(a)$; $y_{2k} = F^{-k}(c)$; $y_{2k+1} = F^{-k}(b)$.

(1) Let $f_0 : [x_1, x_0] \to [r, y_0]$ be a strictly decreasing continuous function with $f_0(x_1) = y_0$ and $f_0(x_0) = r$; and let $g_0 : [y_0, y_1] \to [x_0, r]$ be a strictly decreasing continuous function with $g_0(y_0) = r$ and $g_0(y_1) = x_0$.

(2) For $i \geq 1$, we define successively $f_i : [x_{i+1}, x_i] \to [y_{i-1}, y_i]$ and $g_i : [y_i, y_{i+1}] \to [x_i, x_{i-1}]$ by $g_i = f_{i-1}^{-1} \circ F$ and $f_i = g_{i-1}^{-1} \circ F$.

(3) We define $f : I \to I$ by setting $f([x_0, y_0]) = \{r\}$, $f(1) = 0$, $f(0) = 1$, and for each $i \geq 0$, $f|[x_{i+1}, x_i] = f_i$, $f|[y_i, y_{i+1}] = g_i$. It is easy to verify that f is an 2-order decreasing iterative root of F.

Lemma 2.4 *Suppose $F \in \mathcal{F}_{ab}, F(a) = r \in (a,b), F(0) > 0$ and $F(1) < 1$. Let m_0 and m_1 be as in Lemma 2.2, and $n \geq 1$ be an integer. Then F has an $2n$-order decreasing iterative root if $m_0 = m_1 + 1$ and one of the following two conditions holds: (1^0) $F^{m_0}(0) > a$; (2^0) $F^{m_0}(0) = a$, $F^{m_1}(1) = b$ and $n = 1$.*

Proof Set $m = m_0$, then $m_1 = m - 1$.

(1) Suppose that (1^0) holds.

(1.1) Assume $F^{m-1}(1) < b$. Choose two sequences $\{\alpha_i\}_{i=0}^{3n-1}$ and $\{\beta_i\}_{i=0}^{3n-1}$ satisfying $a = \alpha_{3n-1} < \alpha_{3n-2} < \cdots < \alpha_1 < \alpha_0 = F^m(0)$, and $r < \beta_0 < \beta_1 < \cdots < \beta_{3n-2} = F^{m-1}(1) < \beta_{3n-1} = b$.

For $l = 0, 1, \cdots, m-1$ and $i = 0, 1, \cdots, 3n-1$, put $a_{3nl+i} = F^{-l}(\alpha_i)$, and $a_{3nm} =$

$F^{-m}(\alpha_0)$; for $l = 0, 1, \cdots, m-1$ and $i = 0, 1, \cdots, 3n-2$, put $b_{3nl+i} = F^{-l}(\beta_i)$; for $l = 0, 1, \cdots, m-2$, put $b_{3nl+3n-1} = F^{-l}(\beta_{3n-1})$. Set $a_{-1} = b_{-1} = r$. For $i = 1, 2, \cdots, 3n-1$, let $g_i : [b_{i-1}, b_i] \to [a_{i-1}, a_{i-2}]$ be a strictly decreasing continuous function with $g_i(b_{i-1}) = a_{i-2}$ and $g_i(b_i) = a_{i-1}$. For $i = 2, 3, \cdots, 3n-1$, let $f_i : [a_i, a_{i-1}] \to [b_{i-3}, b_{i-2}]$ be a strictly decreasing continuous function with $f_i(a_i) = b_{i-2}$ and $f_i(a_{i-1}) = b_{i-3}$. Let $f_1 : [a_1, a_0] \to [a_1, a_0]$ be the constant function, where the constant is r. For $j = 0, 1, \cdots, 3n(m-1)-2$, we define successively $g_{3n+j} = f^{-1}_{3n-1+j} \circ g^{-1}_{3(n-1)+j} \circ \cdots \circ g^{-1}_{6+j} \circ f^{-1}_{5+j} \circ g^{-1}_{3+j} \circ f^{-1}_{2+j} \circ F$,and $f_{3n+j} = g^{-1}_{3n-2+j} \circ f^{-1}_{3(n-1)+j} \circ \cdots \circ f^{-1}_{6+j} \circ g^{-1}_{4+j} \circ f^{-1}_{3+j} \circ g^{-1}_{1+j} \circ F$. We can define f_{3nm-1} and f_{3nm} in the same way as above.

Define $f : I \to I$ by $f([a_0, b_0]) = \{r\}$; for $i = 1, 2, \cdots, 3nm$, $f|[a_i, a_{i-1}] = f_i$; for $i = 1, 2, \cdots, 3nm-2$, $f|[b_{i-1}, b_i] = g_i$. It is easy to verify that f is a $2n$-order decreasing iterative root of F.

(1.2) Assume $F^{m-1}(1) = b$. Choose two sequences of real numbers $\{\alpha_i\}_{i=0}^{3n-1}$ and $\{\beta_i\}_{i=0}^{3n-1}$ satisfying $a = \alpha_{3n-1} < \alpha_{3n-2} < \cdots < \alpha_1 < \alpha_0 = F^m(0)$, and $F(a) < \beta_0 < \beta_1 < \cdots < \beta_{3n-2} < \beta_{3n-1} = b$.

In the same way we can construct a $2n$-order decreasing iterative root of F.

(2) Suppose (2^0) hold. For $i = 0, 1, \cdots, m$, put $a_i = F^{-i}(a)$; for $i = 0, 1, \cdots, m-1$, put $b_i = F^{-i}(b)$. Let $g_0 : [r, b_0] \to [a_0, r]$ be a strictly decreasing continuous function satisfying $g_0(r) = r$ and $g_0(b_0) = a_0$. Let $f_0 : [a_0, r] \to [a_0, r]$ be the constant function, where the constant is r. For $i = 0, 1, \cdots, m-1$, we define successively $f_i = g^{-1}_{i-1} \circ F$ and $g_i = f^{-1}_{i-1} \circ F$. Let $f_m = g^{-1}_{m-1} \circ F$. We define $f : I \to I$ by setting $f|[a_0, r] = f_0; f|[r, b_0] = g_0$; for $i = 1, 2, \cdots, m$, $f|[a_i, a_{i-1}] = f_i$; for $i = 1, 2, \cdots, m-1$, $f|[b_{i-1}, b_i] = g_i$. It is easy to verify that f is a 2-order decreasing iterative root of F. □

By a similar argument, we have

Lemma 2.4* *Suppose $F \in \mathcal{F}_{ab}, F(a) = r \in (a, b), F(0) > 0$ and $F(1) < 1$. Let m_0 and m_1 be as in Lemma 2.2, and $n \geq 1$ be an integer. Then F has an $2n$-order decreasing iterative root if $m_1 = m_0 + 1$ and $F^{m_1}(1) < b$.*

Lemma 2.5 *Suppose $F \in \mathcal{F}_{ab}, F(a) = r \in (a, b), F(0) > 0$ and $F(1) < 1$. Let m_0 and m_1 be as in Lemma 2.2, and $n \geq 2$ be an even. If $m_0 = m_1$ then F has an $2n$-order decreasing iterative root.*

Proof Set $m = m_0$. It is obvious that $r \in Fix(F)$.

(1) Suppose $F^m(0) > a$ and $F^m(1) < b$. Choose two sequences of real numbers $\{\alpha_i\}_{i=0}^{n-1}$ and $\{\beta_i\}_{i=0}^{n-1}$ satisfying $a = \alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_1 < \alpha_0 = F^m(0)$, and $F^m(1) = \beta_0 < \beta_1 < \cdots < \beta_{n-2} < \beta_{n-1} = b$.

For $l = 0, 1, \cdots m$; $i = 0, 1, \cdots, n-1$ put $a_{nl+i} = F^{-l}(\alpha_i)$ and $b_{nl+i} = F^{-l}(\beta_i)$. Put $a_{-1} = b_{-1} = r$. For $i = 1, 2, \cdots, n-1$, let $f_i : [a_i, a_{i-1}] \to [b_{i-2}, b_{i-1}]$ be a strictly decreasing continuous function satisfying $f_i(a_i) = b_{i-1}$ and $f_i(a_{i-1}) = b_{i-2}$; $g_i : [b_{i-1}, b_i] \to [a_{i-1}, a_{i-2}]$ be a strictly decreasing continuous function satisfying $g_i(b_{i-1}) = a_{i-2}$ and $g_i(b_i) = a_{i-1}$. For $j = 0, 1, \cdots, n(m-1)$, we define successively $f_{n+j} = g^{-1}_{n-1+j} \circ f^{-1}_{n-2+j} \circ \cdots \circ f^{-1}_{2+j} \circ g^{-1}_{1+j} \circ F$, and $g_{n+j} = f^{-1}_{n-1+j} \circ g^{-1}_{n-2+j} \circ \cdots \circ g^{-1}_{2+j} \circ f^{-1}_{1+j} \circ F$.

We define $f : I \to I$ by setting $f|[a_i, a_{i-1}] = f_i, f|[b_{i-1}, b_i] = g_i$ for $i = 1, 2, \cdots, nm$, and $f([a_0, b_0]) = \{r\}$. It is easy to verify that f is an n-order decreasing iterative root of F.

(2.1) Suppose $F^m(0) = a$ and $F^m(1) < b$. Choose two sequences of real numbers $\{\alpha_i\}_{i=0}^{n-1}$ and $\{\beta_i\}_{i=0}^{n-1}$ satisfying $a = \alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_1 < \alpha_0 < r < \beta_0 < \beta_1 < \cdots < \beta_{n-2} = F^m(1) < \beta_{n-1} = b$.

(2.2) Suppose $F^m(0) > a$ and $F^m(1) = b$. Choose two sequences of real numbers $\{\alpha_i\}_{i=0}^{n-1}$ and $\{\beta_i\}_{i=0}^{n-1}$ satisfying $a = \alpha_{n-1} < F^m(0) = \alpha_{n-2} < \cdots < \alpha_1 < \alpha_0 < r < \beta_0 < \beta_1 < \cdots < \beta_{n-2} < \beta_{n-1} = b$.

(2.3) Suppose $F^m(0) = a$ and $F^m(1) = b$. Choose two sequences of real numbers $\{\alpha_i\}_{i=0}^{n-1}$ and $\{\beta_i\}_{i=0}^{n-1}$ satisfying $a = \alpha_{n-1} < \alpha_{n-2} < \cdots < \alpha_1 < \alpha_0 < r < \beta_0 < \beta_1 < \cdots < \beta_{n-2} < \beta_{n-1} = b$.

For the cases (2.1)–(2.3), we can construct an n-order decreasing iterative roots of F in the same way.

The Proof of Theorem 2 The sufficiency follows immediately from Lemmas 2.3–2.5 and 2.4*. The necessity follows from Lemmas 2.1 and 2.2 and the following claim: Suppose $F(a) \in (a, b)$ and one of the following two conditions holds: (c_1) $F(0) > 0$ and $F(1) = 1$; (c_2) $F(0) = 0$ and $F(1) < 1$, then F has no decreasing iterative root.

Assume, on the contrary, that the claim doesn't hold. Without loss generality, we may suppose that (c_1) holds and $F(a) \in (a, b)$. If F has a decreasing iterative root f, then it follows from Lemmas 1.1 and 1.2 that $f(0) = 1$ and $f(1) = 0$. Thus $F(0) = 0$, which is a contradiction. Hence F has no decreasing iterative root. □

3. Necessary and sufficient conditions for $F \in \Phi_{ab} \backslash \mathcal{F}_{ab}$ having n-order decreasing iterative roots

The Proof of Theorem 3 The sufficiency: Evidently, the cardinality of $E \geq 5$. According to Lemma 1.5, Remark 1.1 and Hardy-Böedwadt theorem (see [2]), it suffices to show that F has a 2-order descreasing iterative root.

(1) Suppose $F(0) = 0$, and $F(1) = 1$. We may assume $\mathrm{Fix}(F) = \{x_{-k}, \cdots, x_{-2}, x_{-1}, y_0, x_1, x_2, \cdots, x_k\}$ with $x_{-k} < x_{-k+1} < \cdots < x_{k-1} < x_k$ (If $Fix(F)$ is a countable set, a similar method can be used). According to Lemma 2.3, $F|[x_{-1}, x_1]$ has a 2-order descreasing iterative root f_0 with $f_0(x_{-1}) = x_1$ and $f_0(x_1) = x_{-1}$. Without loss generality, we may assume $F(x) - x < 0$ for $x \in (x_{-2}, x_{-1})$. Choose arbitrarily two points $a_0 \in (x_{-2}, x_{-1})$ and $b_0 \in (x_1, x_2)$. For each integer l, put $a_l = F^l(a_0)$ and $b_l = F^l(b_0)$. Let $h_0 : [a_1, a_0] \to [b_0, b_1]$ be a strictly decreasing continuous function with $h_0(a_1) = b_1$ and $h_0(a_0) = b_0$.

(1.1) For $j = 0, 1, \cdots$, define successively $g_{-j} : [b_{-(j+1)}, b_{-j}] \to [a_{-j+1}, a_{-j}]$ and $h_{-(j+1)} : [a_{-j}, a_{-(j+1)}] \to [b_{-(j+1)}, b_{-j}]$ by $g_{-j} = h_{-j}^{-1} \circ F$ and $h_{-(j+1)} = g_{-j}^{-1} \circ F$.

(1.2) For $j = 1, 2, \cdots$, define successively $g_j : [b_{j-1}, b_j] \to [a_{j+1}, a_j]$ and $h_j : [a_{j+1}, a_j] \to [b_j, b_{j+1}]$ by $g_j = F \circ h_{j-1}^{-1}$ and $h_j = F \circ g_j^{-1}$.

(1.3) Define $f_{-1} : [x_{-2}, x_{-1}] \to [x_1, x_2]$ by setting $f_{-1}(x_{-1}) = x_1$, $f_{-1}(x_{-2}) = x_2$, and $f_{-1}|[a_{j+1}, a_j] = h_j$ for each integer j. Define $f_1 : [x_1, x_2] \to [x_{-2}, x_{-1}]$ by setting $f_1(x_1) = x_{-1}, f_1(x_2) = x_{-2}$ and $f_1|[b_{j-1}, b_j] = g_j$ for each integer j.

(1.4) Define f_i and f_{-i} in the same way as above for $i = 2, 3, \cdots, k-1$.

Put

$$f(x)=\begin{cases} f_0(x), & \text{if } x\in[x_{-1},x_1];\\ f_{-i}(x), & \text{if } x\in[x_{-(i+1)},x_{-i}),\ i=1,2,\cdots,k-1;\\ f_i(x), & \text{if } x\in(x_i,x_{i+1}],\ i=1,2,\cdots,k-1 \end{cases}$$

It is easy to verify that f is an 2-order decreasing iterative root of F.

(2) Suppose $F(0)>0$ and $F(1)<1$. Choose s,t with $s<0, t>1$. Define F_{st} : $[s,t]\to[s,t]$ by $F_{st}|[0,1]=F$, $F_{st}(s)=s$, $F_{st}(t)=t$, $\mathrm{Fix}(F_{st})=Fix(F)\cup\{s,t\}$ and both $F_{st}|[s,0]$ and $F_{st}|[1,t]$ be strictly increasing. According to the conclusion as above, F_{st} has an 2-order decreasing iterative root $f_{st}:[s,t]\to[s,t]$ such that both $f_{st}|[s,x_{-1}]$ and $f_{st}|[x_1,t]$ are strictly decreasing. Put $f_{01}=f_{st}|[0,1]$, then f_{01} is an 2-order decreasing iterative root of $F=F_{st}|[0,1]$.

The necessity: Suppose f is an n-order decreasing iterative root of F, then n is even and y_0 is the unique fixed point of f.

For each $e\in E_1$, put $D(e)=f(e)$, then $D(e)>f(y_0)=y_0$ and $F\circ D(e)=F\circ f(e)=f\circ F(e)=f(e)=D(e)$ so $D(e)\in E_2$. Thus we obtain an order-reserving one to one map $D:E_1\to E_2$. Aussme $e_1<e_2$ are two consecutive points in E, where either e_1 or e_2 is not y_0. If $F(x)-x$ is positve (or negative) in (e_1,e_2), then $f(y)\in(D(e_2),D(e_1))$ for any $y\in(e_1,e_2)$. Since $F(y)>y$ (or $F(y)<y$), it follows that $F\circ f(y)<f(y)$ $(F\circ f(y)>f(y))$, so $F(x)-x$ is negative (or positive) in $(D(e_2),D(e_1))$. Since $\overline{a}<y_0<\overline{b}$ are adjacent points in E, $F(x)-x$ is positive in $(\overline{a},y_0)$ if and only if $F(x)-x$ is negative in $(y_0,\overline{b})$.

If $f(0)=1$ and $f(1)=0$, then $F(0)=0$ and $F(1)=1$; If $f(0)<1$ then $F(0)=f^n(0)>f^{n-1}(1)\geq 0$, and $F(1)=f^n(1)\leq f(0)<1$; By a similar argument, it holds that $(F(0)-0)(F(1)-1)<0$ if $f(1)>0$. Hence either $(F(0)-0)(F(1))-1)<0$ or $F(0)+1-F(0)=0$. □

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References:

[1] KUCZMA M. *Fractional iteration of differentiable functions* [J]. Ann. Polon. Math., 1969/1970, **22**: 217–227.

[2] ZHANG Jing-zhong and YANG Lu. *Discussion on iterative roots of piecewise monotone function* [J]. In Chinese, Acta Math. Sinica, 1983, **26**(4): 398–412.

[3] MAI Jie-hua. *Conditions for self-homeomorphisms on S^1 having N-order iterative roots* [J]. In Chinese, Acta Math. Sinica, 1987, **30**(2): 280–283.

[4] SIMON K. *On the Iterates of Continuous Functions* [M]. ECIT 89, 343–346.

[5] CHOCZEWSKI B and KUCZMA M., *On Iterative Roots of Polynomials* [M]. ECIT 91, 59–67.

[6] ZHANG Jing-zhong, YANG Lu and ZHANG Wei-nian. *Some Advances on Functional Equations* [J]. Advances in Mathematics, 1995, **24**(5): 385–405.

区间上一类连续自映射的单调迭代根

刘 新 和[1,2]
(1. 广西大学数学与信息科学系，南宁 530004; 2. 汕头大学数学系，广东 515063)

摘 要: 设 $I=[0,1]$, $0<a<b<1$, 记 $\Phi_{ab}\equiv\{F\in C^0(I):\ F|[0,a]$ 和 $F|[b,1]$ 严格单调递增且 F 在 $[a,b]$ 恒取常值 $\}$. 本文讨论了 $F\in\Phi_{ab}$ 有单调迭代根的充要条件.