Journal of Mathematical Research & Exposition
Vol.20, No.4, 498-502, November, 2000

Tensor Coalgebras in Cotringular Hopf Comodule Category *

CEN Jian-miao
(Dept. of Math., Ningbo University, Zhejiang 315211, China)

Abstract: We discuss the Hopf algebra structures of the tensor coalgebras in cotriangular Hopf algebras comodule category.

Key words: cotriangular Hopf algebra; comodule category; tensor coalgebra.

Classification: AMS(1991) 16W30/CLC O153.3

Document code: A **Article ID:** 1000-341X(2000)04-0498-05

In [1], Yu. I. Manin introduced many algebra structures in categories. In this paper, we will discuss the tensor coalgebras in cotriangular Hopf algebras comodule category and give the Hopf algebra structures of the tensor coalgebras in cotriangular Hopf algebras comodule category.

Throughout, $(H,\mu,\eta,\Delta,\varepsilon,S)$ denotes a Hopf algebra over a field K. $\mathcal{M}^H$ denotes the category of right H-comodules. For $M\in \mathrm{Obj}(\mathcal{M}^H)$, the structure map of M is denoted by $\phi_M : M \longrightarrow M\otimes H; m\longmapsto \sum m^{(0)}\otimes m^{(1)}$. For K, define $\phi_K : K\longrightarrow K\otimes H; k\longmapsto k\otimes 1_H$. Then, $K\in \mathrm{Obj}(\mathcal{M}^H)$. For M, $N\in \mathrm{Obj}(\mathcal{M}^H)$, define $\phi_{M\otimes N} : M\otimes N \longrightarrow M\otimes N\otimes H; m\otimes n\longmapsto \sum m^{(0)}\otimes n^{(0)}\otimes m^{(1)}n^{(1)}$. Then, $M\otimes N\in \mathrm{Obj}(\mathcal{M}^H)$.

A Hopf algebra H is called coquasitriangular (or dual quasitriangular, see [2]) if there exists a bilinear form $R : H\otimes H \longrightarrow K$, which is convolution invertible in $\mathrm{Hom}_K(H\otimes H, K)$, such that for all $h,k,l\in H$,

$$\sum R(h_{(1)}\otimes k_{(1)})k_{(2)}h_{(2)} = \sum h_{(1)}k_{(1)}R(h_{(2)}\otimes k_{(2)}); \tag{1}$$

$$R(h\otimes kl) = \sum R(h_{(1)}\otimes l)R(h_{(2)}\otimes k); R(hk\otimes l) = \sum R(h\otimes l_{(1)})R(k\otimes l_{(2)}). \tag{2}$$

Lemma 1[3] *If (H,R) is a coqusitriangular Hopf algebra, then*

$$R(h\otimes 1_H) = \varepsilon(h) = R(1_H\otimes h); \tag{3}$$

*Received date: 1998-02-23
Biography: CEN Jian-miao (1959-),male, born in Cixi county, Zhejiang province, currently a professor at Ningbo University.

$$\sum R(h_{(1)} \otimes k_{(1)})R(h_{(2)} \otimes l_{(1)})R(k_{(2)} \otimes l_{(2)})$$
$$= \sum R(k_{(1)} \otimes l_{(1)})R(h_{(1)} \otimes l_{(2)})R(h_{(2)} \otimes k_{(2)}). \tag{4}$$

Let (H, R) be a coqusitriangular Hopf algebra and M, N right H-comodules. Then,

$$\Psi : M \otimes N \longrightarrow N \otimes M; m \otimes n \longmapsto \sum n^{(0)} \otimes m^{(0)} R(m^{(1)} \otimes n^{(1)}) \tag{5}$$

is a right H-comodule isomorphism. And, $\mathcal{M}^H$ is a braided monoidal category with a braiding Ψ and H is cotriangular if and only if $\mathcal{M}^H$ is symmetrically monoidal (see [2] or [4]). For $V_1, V_2, \cdots, V_n \in \mathrm{Obj}(\mathcal{M}^H)$,

$$\Psi_{ij} : V_1 \otimes \cdots \otimes V_i \otimes \cdots \otimes V_j \otimes \cdots \otimes V_n \longrightarrow V_1 \otimes \cdots \otimes V_j \otimes \cdots \otimes V_i \otimes \cdots \otimes V_n$$

denotes the H-comodule morphism which sends $v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_n$ to $\sum v_1 \otimes \cdots \otimes v_j^{(0)} \otimes \cdots \otimes v_i^{(0)} R(v_i^{(1)} \otimes v_j^{(1)}) \otimes \cdots \otimes v_n$. All Ψ_{ij}'s are called H-transposition. Then, by Theorem 1.16 in [2] and Theorem 10.4.2 in [4], we have the following.

Lemma 2 *Let $V_1, V_2, V_3 \in \mathrm{Obj}(\mathcal{M}^H)$. Then,*

$$\Psi_{1,23} = \Psi_{23}\Psi_{12}, \ \Psi_{12,3} = \Psi_{12}\Psi_{23}, \ \Psi^2 = id, \tag{6}$$

$$\Psi_{12}\Psi_{23}\Psi_{12} = \Psi_{23}\Psi_{12}\Psi_{23}, \tag{7}$$

where $\Psi_{1,23} : V_1 \otimes (V_2 \otimes V_3) \longrightarrow (V_2 \otimes V_3) \otimes V_1; v_1 \otimes (v_2 \otimes v_3) \longmapsto \sum (v_2 \otimes v_3)^{(0)} \otimes v_1^{(0)} R(v_1^{(1)} \otimes (v_2 \otimes v_3)^{(1)})$ and $\Psi_{12,3}$ is similar.

The ideas of an algebra and coalgebra in the category $\mathcal{M}^H$ is just the usual one. But now we use the morphisms in $\mathcal{M}^H$ and H-transposition Ψ_{ij} instead of the linear maps and transposition τ_{ij} respectively. Throughout, all algebra structures are in $\mathcal{M}^H$. We can obtain easily the following lemmas.

Lemma 3 *Let (A, μ_A) and (C, Δ_C) be algebra and coalgebra in $\mathcal{M}^H$ respectively. Then*

$$\Psi(id \otimes \mu_A) = (\mu_A \otimes id)\Psi_{23}\Psi_{12}, \ \Psi(\mu_A \otimes id) = (id \otimes \mu_A)\Psi_{12}\Psi_{23}; \tag{8}$$

$$(id \otimes \Delta_C)\Psi = \Psi_{12}\Psi_{23}(\Delta_C \otimes id), \ (\Delta_C \otimes id)\Psi = \Psi_{23}\Psi_{12}(id \otimes \Delta_C). \tag{9}$$

Lemma 4 *Let $M \in \mathcal{O}\lfloor|(\mathcal{M}^H)$. Write $M_H^* = \mathrm{Com}_H(M, K)$, the set all H-comodule morphisms from M to K. Then,*

(1) M_H^* is a right H-comodule via

$$\varphi_{M_H^*} : M_H^* \longrightarrow M_H^* \otimes H; m^* \longmapsto m^* \otimes g, \tag{10}$$

where g is a group-like element in H.

(2) If $g = 1_H$, then the K-linear injection

$$\rho_M : M_H^* \otimes M_H^* \longrightarrow (M \otimes M)_H^*; \rho(m^* \otimes n^*)(m \otimes n) = m^*(m)n^*(n) \tag{11}$$

is an H-comodule morphism.

(3) Let $N \in \mathrm{Obj}(\mathcal{M}^H)$ and $f : M \longrightarrow N$ be a morphism in $\mathcal{M}^H$. Write

$$f^* : N_H^* \longrightarrow M_H^*; n^* \longmapsto n^* f. \tag{12}$$

Then, f^* is also a morphism in $\mathcal{M}^H$.

Definition 5 *Let $W \in \mathcal{M}^H$. A tensor coalgebra in $\mathcal{M}^H$ on W consists of a coalgebra C in $\mathcal{M}^H$ together with a morphism $p_W : C \longrightarrow W$ of $\mathcal{M}^H$ such that if B is any coalgebra in $\mathcal{M}^H$ and $f : B \longrightarrow M$ any morphism of $\mathcal{M}^H$, then there is an unique coalgebra morphism $g : B \longrightarrow C$ making the following diagram commutative*

$$\begin{array}{ccc} W & \xleftarrow{p_W} & C \\ f\uparrow & & \nearrow_g \\ B & & \end{array}$$

that is, $p_W g = f$.

Obviously, in the sense of isomorphism, the tensor coalgebra on W is unique. Thus, $T^C(W)$ denotes the tensor coalgebra on W. Now, we discuss the existence of $(T^C(W), p_W)$.

Let $W \in \mathcal{M}^H, (T(W), i_W)$ be the tensor algebra on W, $(T(W))_H^\circ$ the dual coalgebra of $T(W)$. Then there exists a morphism $i_W^* i : (T(W))_H^\circ \hookrightarrow^i (T(W))_H^* \longrightarrow^{i_H^*} W_H^*$ of $\mathcal{M}^H$. By the way in [5, 6.4], we can proved similarly the following lemma.

Lemma 6 *(1) $T^C(W)_H^* = (T(W))_H^\circ$.*

(2) Let V be an H-sumcomodule of W. If $T^C(W)$ exists, then $T^C(V)$ exists.

Thus, we have te following results.

Theorem 7 *Let $W \in \mathcal{M}^H$. Then the tensor coalgebra on W exists.*

Proof Since $W \in \mathcal{M}^H, W_H^* \in \mathcal{M}^H$. By the first part of lemma 6 we know that $T^C(W_H^*)_H^*$ exists. Since $\Psi_W : W \longrightarrow (W_H^*)_H^*; \Psi_W(m)(f) = f(m)$ is injective, then $T^C(W)$ exists by (2) of Lemma 6.

We now discuss Hopf algebra structure of $T^C(W)$.

Let $W \in \mathcal{M}^H$ and $(T^C(W), \Delta_{T^C(W)}, \varepsilon_{T^C(W)}, p_W)$ be the tensor coalgebras on W. Set

$$\mu : T^C(W) \otimes T^C(W) \longrightarrow W; x \otimes y \longmapsto \varepsilon_{T^C(W)}(y) p_W(x) + \varepsilon_{T^C(W)}(x) p_W(y); \tag{13}$$

$$\eta : K \longrightarrow W; k \longmapsto 0; \tag{14}$$

$$S : (T^C(W))^{OP} \longrightarrow W; x \longmapsto -p_W(x), \tag{15}$$

where $T^C(W) \otimes T^C(W)$ is a brained tensor product with structure maps, $\Delta_{T^C(W) \otimes T^C(W)} = (\mathrm{id} \otimes \Psi \otimes \mathrm{id})(\Delta_{T^C(W)} \otimes \Delta_{T^C(W)})$ and $\varepsilon_{T^C(W) \otimes T^C(W)} = \varepsilon_{T^C(W)} \otimes \varepsilon_{T^C(W)}$. $(T^C(W))^{OP}$ denotes the brained opposite coalgebra of $T^C(W)$ with structure $\Delta_{(T^C(W))^{OP}} = \Psi \Delta_{T^C(W)}$ and $\varepsilon_{(T^C(W))^{OP}} = \varepsilon_{T^C(W)}$. Then, μ is a morphism in $\mathcal{M}^H$. In fact, for $h \in H, x, y \in$

$T^C(W)$.

$$
\begin{aligned}
(\mu\otimes \mathrm{id})\phi(x\otimes y) &= (\mu\otimes \mathrm{id})(\sum x^{(0)}\otimes y^{(0)}\otimes x^{(1)}y^{(1)})\\
&= \sum(\varepsilon_{T^C(W)}(y^{(0)})p_W(x^{(0)})+\varepsilon_{T^C(W)}(x^{(0)})p_W(y^{(0)}))\otimes x^{(1)}y^{(1)}\\
&= \sum \varepsilon_{T^C(W)}(y)p_W(x^{(0)})\otimes x^{(1)}+\sum \varepsilon_{T^C(W)}(x)p_W(y^{(0)})\otimes y^{(1)}\\
&= \varepsilon_{T^C(W)}(y)\sum p_W(x^{(0)})\otimes x^{(1)}+\varepsilon_{T^C(W)}(x)\sum p_{T^C(W)}(y^{(0)})\otimes y^{(1)}\\
&= \varepsilon_{T^C(W)}(y)\phi p_W(x)+\varepsilon_{T^C(W)}(x)\phi p_W(y)\\
&= \phi(\varepsilon_{T^C(W)}(y)p_W(x)+\varepsilon_{T^C(W)}(y)p_W(y)) = \phi\mu(x\otimes y).
\end{aligned}
$$

Hence, μ is a an H-comodule morphism. It is known easily that η and S are also morphisms in $\mathcal{M}^H$. By the U. M. P. of $T^C(W)$. There exist coalgebra maps

$$\mu_{T^C(W)}: T^C(W)\otimes T^C(W)\longrightarrow T^C(W) \tag{16}$$

$$\eta_{T^C(W)}: K\longrightarrow T^C(W) \tag{17}$$

$$S_{T^C(W)}: (T^C(W))^{OP}\longrightarrow T^C(W) \tag{18}$$

such that $p_W\mu_{T^C(W)}=\mu, p_W\eta_{T^C(W)}=\eta, p_W S_{T^C(W)}=S$ respectively. Therefore, we obtain the Hopf algebra structure of the tensor coalgebra $T^C(W)$. □

Theorem 8 *Let $W\in\mathcal{M}^H$. Then $(T^C(W),\ \mu_{T^C(W)},\ \eta_{T^C(W)},\ \Delta_{T^C(W)},\ \varepsilon_{T^C(W)},\ S_{T^C(W)})$ are all Hopf algebras in $\mathcal{M}^H$ and $S^2_{T^C(W)}=\mathrm{id}_{T^C(W)}$.*

Proof We only prove that $S_{T^C(W)}$ is antipode of $T^C(W)$. Since $S_{T^C(W)}$ is a coalgebra map, for $x\in T^C(W)$, we have

$$
\begin{aligned}
p_W(S_{T^C(W)}*\mathrm{id})(x) &= p_W\mu_{T^C(W)}(\sum S_{T^C(W)}(x_{(1)})\otimes x_{(2)})\\
&= \mu(\sum S_{T^C(W)}(x_{(1)})\otimes x_{(2)})\\
&= \sum(\varepsilon_{T^C(W)}(S_{T^C(W)}(x_{(1)}))p_W(x_{(2)})+\varepsilon_{T^C(W)}(x_{(2)})p_W(S_{T^C(W)}(x_{(1)})))\\
&= \sum(\varepsilon_{(T^C(W))^{OP}}(x_{(1)})p_W(x_{(2)})+S_{T^C(W)}(\varepsilon_{(T^C(W))^{OP}}(x_{(2)})p_W(x_{(1)})))\\
&= p_W(x)+p_W S_{T^C(W)}(x) = p_W(x)+S(x)=0\\
&= \eta(\varepsilon_{T^C(W)}(x)) = p_W\eta_{T^C(W)}\varepsilon_{T^C(W)}(x).
\end{aligned}
$$

It is easy to see that $\eta_{T^C(W)}\varepsilon_{T^C(W)}$ is a coalgebra map. Hence, for the H-comodule map $f=p_W(S_{T^C(W)}*id)$, by the uniqueness of g in Definition 5, $S_{T^C(W)}*id=\eta_{T^C(W)}\varepsilon_{T^C(W)}$. Similarly, we can prove that $\mathrm{id}*S_{T^C(W)}=\eta_{T^C(W)}\,\varepsilon_{T^C(W)}$. Therefore, $S_{T^C(W)}$ is a antipode of $T^C(W)$. □

References:

[1] MANIN Y I. *Quantum group and non-commutative geometry* [M]. CRM, Universite de Montreal, 1988.

[2] MAJID S. *Algebras and Hopf algebras in Braided Categories* [J]. Lecture notes in pure and applied mathematics, 1995, **158**: 55–105.
[3] MAJID S. *Braided Groups* [J]. J. Pure and Applied Algebra, 1993, **86**: 187–221.
[4] MONTGOMERY S. *Hopf Algebras and Their Actions on Rings* [M]. CBMS, 82, Chicago, 1993.
[5] SWEEDLER M E. *Hopf Albegras* [M]. Benjamim, New York, 1969.

三角 Hopf 余模范畴上的张量余代数

岑 建 苗

(宁波大学数学系，浙江 315211)

摘 要: 在三角 Hopf 代数余模范畴上研究张量余代数. 主要给出三角 Hopf 代数余模范畴上的张量余代数的结构.