Journal of Mathematical Research & Exposition
Vol.20, No.4, 529–533, November, 2000

Counting Rooted Near-Triangulations on the Cylinder *

REN Han[1], *LIU Yan-pei*[2]

(1. Dept. of Math., Suzhou University, 215006, China;

2. Dept. of Math., Northern Jiaotong University, Beijing 100044, China)

Abstract: In this paper we present a parametric expression on the enumeration of rooted non-separable near-triangulations on the cylinder which is much related to the maps on the torus.

Key words: map; enufunction; Lagrangian inversion.

Classification: AMS(1991) 05C30/CLC O157.5

Document code: A **Article ID:** 1000-341X(2000)04-0529-05

1. Introduction

Maps here are combinatorial as defined in [2] or [3].Let $M = (\mathcal{X}_{\alpha,\beta}(X), \mathcal{P})$ be a map, $r \in X$ is chosen as the root. Then, $v_r(M), e_r(M)$ and $f_r(M)$ stand for the root-vertex,root-edge, and root-face valency of M respectively.

A triangulation is such a planar map whose faces are all triangles, i.e., the boundary of faces are all 3-gons.If a map has all of its faces triangle but possibly one,the root-face, then it is named planar near-triangulation. Moreover, if a map has only two possible non 3-gon faces which have no common vertices, then it is called near-triangulation on the cylinder. It is clear that such maps are greatly involved with the maps on the torus, a kind of non-planar maps difficult to handle.

Let C be the set of all the rooted non-separable near-triangulations on the cylinder and

$$f_C = \sum_{M \in C} x^{m(M)} y^{s(M)} z^{n(M)}$$

be its enufunction, where x and y indicate the root-face valency and the size of M, while z marks the other possible non 3-gon valency.

In this paper f_C is completely determined, i.e.,

*Received date: 1998-04-01
Foundation item: Supported by the Natural Science Foundation of China (19701002)
Biography: REN Han (1958-), male, born in Wuhan city, Hubei province. Ph.D.

Theorem *The enufunction of all the rooted non-separable near-triangulations on the cylinder can be expressed as*

$$f_C = \sum_{m-5\geq i\geq j\geq 0} \sum_{\substack{m\geq m-1 \\ l\geq 0}} C(m,n)\binom{2l}{l}\eta^{i+2l-j+1}y^{3n-m}z^{m-i-3}x^{j+l+2}$$

where

$$\eta^{i+2l-j+1} = \sum_{s\geq 1}\frac{y^s}{s!}D^{s-1}_{\eta=0}\left\{\frac{(i+2l-j+1)\eta^{i+2l-j}}{(1-2y\eta^2)^s}\right\},$$

$$C(m,n) = \frac{2^{n-m+2}(2m-3)!(3n-m-1)!}{(2n)!(n-m+1)!(m-2)!(m-2)!}.$$

2. Decomposition Lemmas

In this section we are concentrated on establishing a functional equation satisfied by f_C.

Let $M \in C$, the removing of the root-edge will lead to three possible situations as shown in Fig.1.

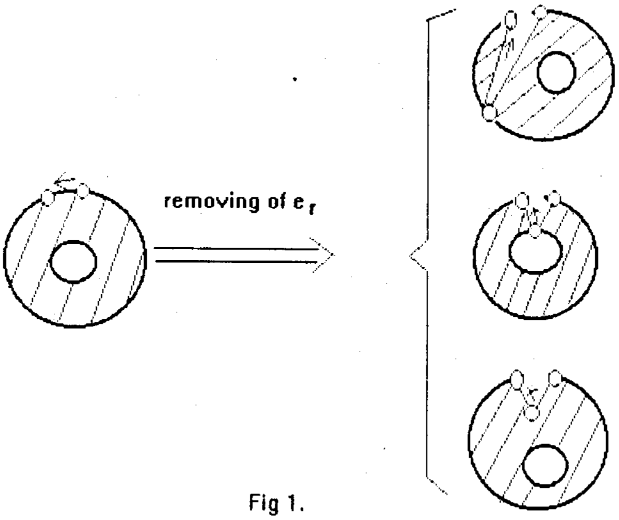

Fig 1.

Thus, C may be divided into three parts:

$$C = C_1 + C_2 + C_3$$

where

$\mathcal{C}_1 = \{M | M - e_r(M)$ is separable$\}$;
$\mathcal{C}_2 = \{M|$ the possible two non 3 – gons of M – e$_r$(M) have a common vertex$\}$;
$\mathcal{C}_3 = \{M | M - e_r(M) \in \mathcal{C}\}$.

Lemma 1 *Let $\mathcal{C}_{\langle i \rangle} = \{M - e_r | M \in \mathcal{C}_i\}, i = 1, 3$. Then*

$$\mathcal{C}_{\langle 1 \rangle} = \mathcal{T}_{ns} \odot \mathcal{C} + \mathcal{C} \odot \mathcal{T}_{ns}; \quad \mathcal{C}_{\langle 3 \rangle} = \mathcal{C} - \mathcal{C}(2)$$

where $\mathcal{T}_{ns}$ is the set of all the rooted nonseparable near-triangulations on the plane; "$\odot$" is the 1v-production defined in [2] and $\mathcal{C}(2)$ is the set of maps in $\mathcal{C}$ with the root-face valency 2.

Proof One may see that the Lemma follows immediately from the definitions of $\mathcal{C}_1$ and $\mathcal{C}_3$. □

Next, we analyse the maps in $\mathcal{C}_{\langle 2 \rangle}$, where $\mathcal{C}_{\langle 2 \rangle} = \{M - e_r(M) | M \in \mathcal{C}_2\}$. For any map M in $\mathcal{C}_{\langle 2 \rangle}$, the root-vertex, $v_r(M)$, is lying on the common border of the only two possible nontriangular faces. The cutting of $v_r(M)$ into v_r and v_r' will lead to a map M' in $\mathcal{T}_{ns}$ with its root face-valency no less than 5. On the other hand, for each map M' in $\mathcal{T}_{ns}$ with $f_r(M') \geq 5$, chosing a reference vertex other than $v_r(M')$ and identifying it with the root-vertex will result in a map in $\mathcal{C}_{\langle 2 \rangle}$. Thus,

$$\begin{aligned} f_2 &= \frac{y}{x} \sum_{M \in \mathcal{T}_{ns}(\geq 5)} \sum_{i=3}^{m(M)-2} x^i y^{s(M)} z^{m(M)-i} \\ &= \frac{y}{(xz)^2(z-x)} (x^4 f_{ns(\geq 5)}(z, y) - z^4 f_{ns(\geq 5)}(x, y)), \end{aligned} \tag{1}$$

where $\mathcal{T}_{ns}(\geq 5) = \mathcal{T}_{ns} - \{M | f_r(M) \leq 4\}$ and $f_{ns(\geq 5)}(x, y)$ is its enufunction.

Recall that(from [1])

$$f_{ns(\geq 5)}(x, y) = \sum_{\substack{m \geq 5 \\ n \geq m-1}} C(m, n) y^{3n-m} x^m,$$

in which

$$C(m, n) = \frac{2^{n-m+2}(2m-3)!(3n-m-1)!}{(2n)!(n-m+1)!(m-2)!(m-2)!}.$$

Substitute this into (1), we have

$$\begin{aligned} f_2(x, y, z) &= \frac{y}{(xz)^2(z-x)} (x^4 f_{ns(\geq 5)}(z, y) - z^4 f_{ns(\geq 5)}(x, y)) \\ &= \sum_{\substack{m \geq 5 \\ n \geq m-1}} \sum_{i=0}^{m-5} C(m, n) y^{3n-m+1} x^{i+2} z^{m-i-3}. \end{aligned}$$

By Lemma 1 and the above relation,

$$f_C = 2\frac{y}{x} f_{ns} f_C + f_2 + \frac{y}{x}(f_C - x^2 F_2) \tag{2}$$

where F_2 is the enufunction of the maps in C with root-face valency 2.

3. Determination of $f_C(x, y, z)$

In this section, we shall determine $f_C(x, y, z)$ completely.
The equation (2) may be rewritten as

$$(x - y - 2y f_{ns}) f_C = x f_2 - y x^2 F_2. \tag{3}$$

Suppose $x = \eta$ be the double root of $\delta = (y - x)^2 - 4y(x^3 y - y x^2 F_2^0)$. Then by [1] we have $x - y - 2y f_{ns} = -\sqrt{\delta}$ and

$$y \eta F_2 = f_2|_{x=\eta}.$$

Let $\delta = y^2(1 - \frac{x}{\eta})^2(1 - vx)$. Then

$$y(\eta - x)\sqrt{1 - vx} f_C = x(x f_2(\eta, y) - \eta f_2(x, y))$$

After simplifying, we have

$$\begin{aligned} f_C &= \frac{1}{\sqrt{1 - vx}} \sum_{\substack{m \geq 5 \\ n \geq m-1}} \sum_{i=0}^{m-5} \sum_{j=0}^{i} C(m, n) \eta^{i-j+1} y^{3n-m} x^{j+2} z^{m-i-3} \\ &= \sum_{m-5 \geq i \geq j \geq 0} \sum_{\substack{n \geq m-1 \\ l \geq 0}} \frac{C(m, n)}{4^l} \binom{2l}{l} v^l \eta^{i-j+1} y^{3n-m} z^{m-i-3} x^{j+l+2}. \end{aligned}$$

Since $v = 4\eta^2$,

$$f_C = \sum_{m-5 \geq i \geq j \geq 0} \sum_{\substack{m \geq m-1 \\ l \geq 0}} C(m, n) \binom{2l}{l} \eta^{i+2l-j+1} y^{3n-m} z^{m-i-3} x^{j+l+2}.$$

Applying Lagrangian inversion to $\eta = \frac{y}{1 - 2y\eta^2}$ and $\eta^{i+2l-j+1}$, we derive

$$\eta^{i+2l-j+1} = \sum_{s \geq 1} \frac{y^s}{s!} D_{\eta=0}^{s-1} \left\{ \frac{(i + 2l - j + 1)\eta^{i+2l-j}}{(1 - 2y\eta^2)^s} \right\}.$$

This completes the proof of the main result.

References:

[1] DONG Feng-ming and LIU Yan-pei. *Counting rooted near-triangulations on the sphere* [J]. Discrete Math., 1993, **123**: 35–45.

[2] LIU Yan-pei. *Enumerative Theory of Maps* [M]. Kluwer, Boston, 1998.
[3] LIU Yan-pei. *Embeddibility in Graphs* [M]. Kluwer Academic Publishers, 1995.
[4] TUTTE W T. *A census of planar maps* [J]. Canada. J. Math., 1963, **15**: 249–271.
[5] TUTTE W T. *Counting rooted triangulations* [J]. Ann. Discrete Math., 1982, **12**: 243–253.

柱面上有根近三角剖分的计数

任 韩[1], 刘彦佩[2]

(1. 苏州大学数学系， 215006; 2. 北方交通大学数学系，北京 100044)

摘 要: 柱面上的三角剖分是一类与环面上的地图紧密相关的地图. 本文提供了一个计算柱面上有根近三角剖分的具有三个变量的精确公式.