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# The Structure of Certain $K_2O_F$ *

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**Abstract:** In this paper, we investigate the structure of $K_2O_F$ for $F=\mathbb{Q}(\sqrt{d}), d\equiv -3 \bmod 9$ and $d\neq -3$. We find the element of order 3 of $K_2O_F$ for $F=\mathbb{Q}(\sqrt{-21})$ and generated elements of $K_2O_F\cong \mathbb{Z}/(2)\oplus\mathbb{Z}/(8)\oplus\mathbb{Z}/(3)$ for $F=\mathbb{Q}(\sqrt{15})$. We get the property of $\Re_2F$, which develops a Tate and Bass's theorem, and give the structure of $K_2O_F$ for $F=\mathbb{Q}(\sqrt{29})$ and the presentation relations of $\mathrm{SL}_n(O_F)(n\geq 3)$.

**Key words:** $K_2$-group; Hilbert symbol; Dennis-Stein symbol.

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## 1. Prerequisites

Let $O_F$ be the ring of integers of a number field $F$. Many papers [4][5] have given explicit finite presentation of certain $K_2O_F$ and $\mathrm{SL}_n(O_F)(n\geq 3)$, and these results are about small size of certain $K_2O_F$.

D.Quillen proved the exact sequence

$$0\longrightarrow K_2O_F\longrightarrow K_2F\stackrel{\tau}{\longrightarrow}\coprod_{v\text{ fin.}}\overline{F}^*\longrightarrow 0, \tag{1.1}$$

where the sum is extended over all finite places $v$ of the number field $F$, and the homomorphism $\tau$ is defined by the tame symbols

$$\{a,b\}\mapsto (a,b)_v=(-1)^{v(a)v(b)}a^{v(b)}b^{-v(a)}(\bmod v).$$

C.Moore got the exact sequence

$$0\longrightarrow \Re_2F\longrightarrow K_2F\stackrel{\eta}{\longrightarrow}\coprod_{v\text{ non }C.}\mu_v\longrightarrow\mu\longrightarrow 0, \tag{1.2}$$

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where the sum is extended over all non-complex places $v$ of $F$, $\mu$ (resp. $\mu_v$) is the group of unit roots in $F$ (the completion field $F_v$), the homomorphism $\eta$ is defined by the Hilbert symbols: $\{a,b\} \mapsto [a,b]_v$. It is well-known that, for any finite place $v$, $[a,b]_v^{m_v/Nv-1} = (a,b)_v$ with $m_v = |\eta_v|$. J.Browkin have described the structure of $K_2O_F/\Re_2F$. When $F=\mathbb{Q}(\sqrt{d}), d \equiv -3 \pmod 9$, and $d \neq -3$, $K_2O_F/\Re_2F$ has an element of order 3.

Put $\Delta = \{a \in F^* | \{-1,a\} = 1\}$. J.Tate[9] proved that $[\Delta : F^{*2}] = 2^{1+r_2}$ for every algebra number fields, where $r_2$ is the number of complex valuations of $F$. Consequently $\Delta = F^{*2} \cup 2F^{*2}$ for every totally real field not containing $\sqrt{2}$.

Assume that $F$ is complete under a discrete valuation $v$ with finite residue class field $\overline{F}$ and $|\overline{F}| = p^f$ where $p$ is prime. Let $e = v(p)$ be the ramification index of $F$. When $F$ contains the $p$-th roots of unity, let $e_0 = e/(p-1)$. Any element $\delta$ of $U_{pe_0}$ ($a \in U_{pe_0}, v(1-a) \geq pe_0$), which does not have a $p$-th root in $U_{e_0}$, will be called a *distinguished unit* of $F$. if $\pi$ is a prime element and $\delta$ is a distinguished unit, then $\{\pi,\delta\}$ is a generated element of Cyclic group $K_2F/(K_2F)^p$ and $[\pi,\delta]_v \neq 1$ (see [7] Appendix).

## 2. Elements of order 3

By [1], we know that

$$|K_2O_F/\Re_2F| = \begin{cases} 3\cdot 2^i, & \text{for } d\equiv -3 \bmod 9,\ d\neq -3, \\ 2^j, & \text{otherwise,} \end{cases}$$

and the element of order 3 of $K_2O_F/\Re_2F$ is determined by the element $\alpha \in K_2O_F$ with $\eta_v(\alpha) \neq 1, v|3$. Therefore, we get

**Lemma 2.1** *Let $d \equiv -3$ mod 9, $d \neq -3$, and $F = \mathbb{Q}(\sqrt{d})$. If $\alpha \in K_2O_F$ and $\eta_v(\alpha), v|3$, is a element of order 3 in $\mu_v$. Then the order of $\overline{\alpha} \in K_2O_F/\Re_2F$ is divided by 3.*

In fact, such element can be found by a distinguished unit in the completion field $F_v, v|3$ (see [7]).

**Example 1** $F = \mathbb{Q}(\sqrt{6})$. Since

$$\begin{aligned}\{6,-8+3\sqrt{6}\} &= \{6,-(\sqrt{6}-2)(\sqrt{6}-1)\} = \{\sqrt{6},2-\sqrt{6}\}^2 \\ &= \{2,2-\sqrt{6}\}^2\{\sqrt{6},2\}^2 = \{2,5-2\sqrt{6}\},\end{aligned}$$

it is obvious that $\{6,-8+3\sqrt{6}\} = \{2,5-2\sqrt{6}\} \in K_2O_F$; on the other hand, $-8+3\sqrt{6} = 1+3\sqrt{6}-9$ is a distinguished unit in $F_v, v|3$, so $[5-2\sqrt{6},2]_v$ is an element of order 3. Let $\varepsilon = 5-2\sqrt{6}, (1-\varepsilon)^2 = 8\varepsilon$,

$$\{\varepsilon,2\}^3 = \{\varepsilon,-8\varepsilon\} = \{\varepsilon,-(1-\varepsilon)^2\} = \{\varepsilon,-1\} = \{8\varepsilon,-1\} = 1.$$

Hence $\{6,-8+3\sqrt{6}\} = \{2,\varepsilon\}$ is an element of order 3 of $K_2O_F$.

**Remark 2.2** In [3], Dennis and Stein used another way to prove that $\{5-\sqrt{6},2\} \neq 1$. But this way is more direct and convenient.

In the following, we use the way to find elements of order 3 in $F = \mathbb{Q}(\sqrt{-21})$ and $F = \mathbb{Q}(\sqrt{15})$.

**Example 2** $F=\mathbb{Q}(\sqrt{-21})$. Since $-21\equiv -3 \bmod 9$, $K_2O_F/\Re_2F$ has a element of order 3. $28+6\sqrt{-21}=1+6\sqrt{-21}+27$ is a distinguished unit in $F_v, v|3$, $[-21, 28+6\sqrt{-21}]_v$ is an element of order 3. It is obvious that $\{-21, 28+6\sqrt{-21}\}^2=\{-21, -(\sqrt{-21}-7)(\sqrt{-21}+1)\}^2=\{-21,\sqrt{-21}-7\}^2\in K_2O_F$ and its order is divides by 3. We will compute $\{-21,\sqrt{-21}-7\}^2$ of order.

$$\{-21,\sqrt{-21}-7\}=\{\sqrt{-21},-21(1+\sqrt{-21}/3)\}^2=\{3,\sqrt{-21}-2\},$$
$$\{-21,\sqrt{-21}-7\}=\{\sqrt{-21},14(2-\sqrt{-21})\}=\{\sqrt{-21},28\}\{2,\sqrt{-21}-2\},$$
$$\{-21,\sqrt{-21}-7\}^2=\{7,-1\}\{4,\sqrt{-21}-2\}=\{4,\sqrt{-21}-2\},$$

where $\{7,-1\}=1$ by [8]. On the other hand,

$$\begin{aligned}\{-2100,\sqrt{-21}-2\}&=\{(\sqrt{-21}+3)(\sqrt{-21}+7),\sqrt{-21}-2\}^2\\&=\{6(\sqrt{-21}-2),\sqrt{-21}-2\}\{7(-3+\sqrt{-21})/\sqrt{-21},\sqrt{-21}-2\}^2\\&=\{-6,\sqrt{-21}-2\}\{-49/21,\sqrt{-21}-2\}=\{14,\sqrt{-21}-2\},\end{aligned}$$

so $1=\{-150,\sqrt{-21}-2\}=\{-6(\sqrt{-21}+2)(2-\sqrt{-21}),\sqrt{-21}-2\}=\{-6(\sqrt{-21}+2),\sqrt{-21}-2\}$, Also

$$1=\{(\sqrt{-21}+2)/4,(2-\sqrt{-21})/4\}=\{\sqrt{-21}+2,2-\sqrt{-21}\}\{2-\sqrt{-21},4\}\{4,\sqrt{-21}+2\},$$
$$\{\sqrt{-21}+2,2-\sqrt{-21}\}=\{4,(2-\sqrt{-21})/(2+\sqrt{-21})\}=\{4,2-\sqrt{-21}\}^2.$$

Hence, we get

$$\{4,\sqrt{-21}-2\}^3=\{\sqrt{-21}+2,2-\sqrt{-21}\}\{4,\sqrt{-21}-2\}=\{-1,-1\}\{28,\sqrt{-21}\}.$$

From the most preceding three equality,

$$\begin{aligned}&\{28,\sqrt{-21}\}=\{6/9,\sqrt{-21}-2\}=\{6(2-\sqrt{-21})/9,\sqrt{-21}-2\}\\&\quad=\{(\sqrt{-21}+3)/3,\sqrt{-21}-2\}^2\{-1,\sqrt{-21}-2\},\\&1=\{\sqrt{-21}-7,-1\}^2=\{14(2-\sqrt{-21}),-1\}=\{7,-1\}\{2-\sqrt{-21},-1\}\\&\quad=\{2-\sqrt{-21},-1\},\\&\{-1,-1\}\{28,\sqrt{-21}\}=\{(\sqrt{-21}+3)/3,\sqrt{-21}-2\}^2.\end{aligned}$$

Let $S=\{v=(5,\sqrt{-21}-2)|\tau_v(\{(\sqrt{-21}+3)/3,\sqrt{-21}-2\})=-1)$, for all finite places $v\}$. But the equation $x^2+21y^2=\varepsilon 5z^2, \varepsilon=1, or 2$, have not solution in $\mathbb{Z}$, Hence $\{-1,-1\}\{28,\sqrt{-21}\}\neq 1$, so $\{4,\sqrt{-21}-2\}$ is an element of order 6 in $K_2O_F$.

**Example 3** Let $F=\mathbb{Q}(\sqrt{15})$. Since $d=15\equiv -3 \bmod 9$, $K_2O_F/\Re_2F$ has an element of order 3. Also $10+3\sqrt{15}=1+3\sqrt{15}+9$ is a distinguished unit in $F_v, v|3$, $\{15,2\}=\{2,15\}^3$, $\{15,20+6\sqrt{15}\}=\{15,(\sqrt{15}+1)(\sqrt{15}+5)\}=\{3,(\sqrt{15}+3)/3\}^2=\{3,\sqrt{15}+4\}\in K_2O_F$. Therefore $[15,20+6\sqrt{15}]_v=[15,10+3\sqrt{15}]_v\in\mu_v$ is an element of order 3, and $3|o(\{3,4+\sqrt{15}\})$. Let $\epsilon=4+\sqrt{15}, \varepsilon^2+\varepsilon+1=9\varepsilon$,

$$\{3,\varepsilon\}^6=\{9,\varepsilon\}^3=\{9\varepsilon,\varepsilon\}^3\{-1,\varepsilon\}=\{\frac{1-\varepsilon^3}{1-\varepsilon},\varepsilon\}^3\{-1,\varepsilon\}=\{-1,\varepsilon\}.$$

Hence $o(\{3,\varepsilon\} = 12$ and $\{-3,\varepsilon-1\} \in K_2O_F$ is an element of order 24 by $\{-3,\varepsilon-1\}^2 = \{3,\varepsilon\}$. By Birch-Tate conjecture, $W_2(F)\zeta_F(-1) = 48$ for $F = \mathbb{Q}(\sqrt{15})$ and $15 = 3 \cdot 5$. We get

**Theorem 2.3** *$F = \mathbb{Q}(\sqrt{15})$, then $K_2O_F \cong \mathbb{Z}/(2) \oplus \mathbb{Z}/(8) \oplus \mathbb{Z}/(3)$, where $\{-1,-1\}$ is a generated element of $\mathbb{Z}/(2)$ and $\{-3, 3+\sqrt{15}\}$ is a generated element of $\mathbb{Z}/(8) \oplus \mathbb{Z}/(3)$.*

## 3. $\Re_2 F$

For quadratic number fields, we know the structure of $K_2O_F/\Re_2F$. In the section, we discuss the property of $\Re_2F$ and find the element of $\Re_2F$.

**Theorem 3.1** *If $\{a,b\} \in \Re_2F$, $m$ is a squarefree integer, then $\{a,b\} \in (K_2O_F)^m$, moreover $a$ is a norm of the ring $F[x]/(x^m - b)$.*

**Proof** Let $o(\{a,b\} = n$. Without loss of generality, suppose that there is a prime $p$ such that $p|m, p|n$, we will prove $\{a,b\} \in (K_2F)^p$.

Since $\{a,b\} \in \Re_2F$, for any finite valuation $v$, $[a,b]_v = 1$. By C.Moore theorem [7], $\{a,b\} \in (K_2F_v)^{m_v}$, which is a divisible group. Hence $\{a,b\} \in (K_2F_v)^{m_v p} \subset (K_2F_v)^p$, so $a$ is a norm an element of the ring $F_v/(x^p - b)$.

If $\xi_p \in F$ is a primitive p-th unit root, then $F(\sqrt[p]{b})/F$ is a cyclic extension. By Hasse theorem [6], we know that $a$ is a norm of a element of the field $F[x]/(x^p - b)$, so $\{a,b\} \in (K_2F)^p$.

If $\xi \notin F$, take $E = F(\xi_p)$. Since $\{a,b\} \in (K_2F_v)^p, \{a,b\} \in (K_2E_{v'})^p$, where $v'$ is any extension of place $v$. Without loss of generality, let $\sqrt[p]{b} \notin E$, by the above discussion, we get $a$ is a norm of $E(\sqrt[p]{b})/E$. Let $\gamma \in E(\sqrt[p]{b}) = L, N_{L/E}(\gamma) = a, N_{L/F(\sqrt[p]{b})}(\gamma) = \theta, N_{E/F}(a) = a^s$, where $s = [E:F]$. Set $i,j \in \mathbb{Z}$ with $is + jp = 1$, then $N_{F(\sqrt[p]{b})/F}(a^j\theta^i) = a^{jp+is} = a$. Hence $\{a,b\} \in (K_2F)^p$ by [7], moreover we know $\{a,b\} \in (K_2F)^m$, where $m$ is a squarefree integer.

In the following, we investigate the structure of $K_2O_F$ of $F = \mathbb{Q}(\sqrt{29})$. Hurrelbrink used the Birch-Tate conjecture to compute that $\#\Re_2F = 3$ and $\#K_2O_F = 12$. We will find the element of order 3 in $\Re_2F$. First, we describe two lemma of J.Tate [9].

**Lemma 3.2** *Let $E = F(\xi_p), \Delta = Gal(E/F)$, then there is an exact sequence*

$$(\xi_p \otimes E^*)^\Delta \longrightarrow K_2F \xrightarrow{p} K_2F \longrightarrow K_2F/(K_2F)^p \longrightarrow 0,$$

*where the homomorphism $\gamma$ is defined as:*

$$\begin{array}{ccc} (\xi_p \otimes E^*)^\Delta & \xrightarrow{\gamma} & (K_2F)_p \\ \downarrow & & \downarrow f \\ \xi_p \otimes E^* & \xrightarrow{\gamma_E} & (K_2E)_p \end{array},$$

*where $f$ is a canonical homomorphism and $\gamma_E : \xi_p \otimes a \mapsto \{\xi_p, a\}$.*

**Lemma 3.3** *Let $r_2$ be the number of complex places of $F$. Let $\varepsilon = 1$ if $[F(\xi_p) : F] \le 2$ and $\epsilon = 0$ otherwise. Then $ker\gamma$ is an elementary abelian group of order $p^{r_2+\varepsilon}$.*

**Theorem 3.4** *$F=\mathbb{Q}(\sqrt{29}), K_2O_F\cong\mathbb{Z}/(2)\oplus\mathbb{Z}/(2)\oplus\mathbb{Z}(3)$, where $\{-1,-1\}$ and $\{-1,\varepsilon\},\varepsilon=\frac{\sqrt{29}+5}{2}$, are generated elements of $\mathbb{Z}/(2)\oplus\mathbb{Z}/(2)$, and $\{w,8\}, w=\frac{\sqrt{29}+1}{2}$, is a generated element of $\mathbb{Z}/(3)(\cong\Re_2F)$.*

**Proof** It is clear that $\{-1,-1\},\{-1,\varepsilon\}\in K_2O_F$ and $\{-1,-1\}\neq\{-1,\varepsilon\}$ by $N(\varepsilon)=-1$. Also $\{8,w\}\in K_2O_F$ and $\{8,w\}\in\Re_2F$ for $-w\in F_v^2(v|2)$. Note that $8=w^2-w+1=(w^3+1)/(w+1)$, so $\{8,-w\}^3=1$. In the following, we will verify that $\{8,w\}\neq1$.

Set $E=F(\xi)$, where $\xi$ is a primitive 3-th unit root. In $K_2E$,

$$\{8,-w\}=\{w^2-w+1,-w\}=\{(1+w\xi)(1+w\xi^2),-w\}=\{\xi,\xi+w\}.$$

Suppose that $\{8,-w\}=1$, then $\gamma(\xi\otimes\xi)=1,\gamma(\xi\otimes(\xi+w))=\{8,-w\}=1$. But $\xi\otimes(\xi+w),\xi\otimes\xi\in(\mu_3\otimes E)^{\Delta}$, and $\xi\otimes(\xi+w)\neq\xi\otimes\xi$, which is a contradiction with Lemma 3.3. Hence $\{8,-w\}\neq1$.

Since $1=\{\frac{2}{w+2},\frac{w}{w+2}\}=\{2,w\}\{w+2,2\}\{-w,w+2\},\{-w,w+2\}=\{2,w+2\}\{w,2\}$. Also $\{1-(w+2)^3,w+2\}=\{2w(w+2),w+2\}=\{2,w+2\}\{-w,w+2\}=\{4,w+2\}\{w,2\}$, so $\{8,w+2\}^2=\{4,w+2\}^3=\{8,w\}$. We easily know that $\{8,w+2\}\in\Re_2F$ is an element of order 3.

**Theorem 3.5** *$F=\mathbb{Q}(\sqrt{29})$. Then $SL_n(O_F)(n\geq3)$ has the following presentation:*
*generators: elementary matrices $x_{ij}(t)$ with $t\in O_F$,*
*relations: usual Steinberg relations and in addition only the three relations given by:*

$$h_{12}(-1)=1; h_{12}(-w-2)=h_{12}(-1)h_{12}(w+2);$$

*and*

$$x_{21}(\frac{w(w+2)}{(w+2)^2})x_{12}(-2)x_{21}(-w(w+1)(w+2))x_{12}(\frac{2}{(w+2)^3})=h_{12}(w+2)^3.$$

**Proof** It is well-known that $K_2(n,O_F)\cong K_2O_F$ and $SL_n(O_F)\cong St_n(O_F)/K_2O_F$, where $St_n(O_F)$ is a Steinberg group. We obtain an explicit presentation of $SL_n(O_F)$ in terms of the usual relations of the Steinberg group $St_n(O_F)$ by adding only those relations which correspond to a set of generators of $K_2O_F$. $\{8,w+2\}$ may be presented by Dennis-Stein symbol [3] in $K_2O_F$. Since $(w+2)^3-1=2w(w+1)(w+2)$, take $a=-2, b=-w(w+1)(w+2)$. $\langle a,b\rangle$ looks like $\{8,w+2\}$. Hence we get the presentation of $SL_n(O_F)$.

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# 某些 $K_2O_F$ 的结构

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**摘　要**: 本文研究二次数域 $F=\mathbb{Q}(\sqrt{d})$ 的 $K_2O_F$ 结构，其中 $d\equiv -3 \bmod 9$ 和 $d\neq -3$. 找到了关于 $F=\mathbb{Q}(\sqrt{-21})$ 的 $K_2O_F$ 的 3 阶元和 $F=\mathbb{Q}(\sqrt{15})$ 的 $K_2O_F$ 的生成元. 推广了 Bass 和 Tate 的一个定理和给出了 $F=\mathbb{Q}(\sqrt{29})$ 的 $K_2O_F$ 的结构以及 $\mathrm{SL}_n(O_F)(n\geq 3)$ 的表示关系.