Journal of Mathematical Research & Exposition
Vol.21, No.1, 7–16, February, 2001

Inverse Chain of Inverse Relation *

MA Xin-rong[1], *WANG Tian-ming*[2]
(1. Dept. of Math., Suzhou University, Jiangsu 215006, China;
2. Dept. of Math., Dalian University of Technology, Liaoning 116024, China)

Abstract: This paper is devoted to investigations on the general theory of inverse chain of arbitrary inverse relation with emphasis on the inverse chain of Gould-Hsu inverse and of its q-analogue. Some new identities are obtained under this point of view.

Key words: inverse matrix; inverse chain; identity.

Classification: AMS(1991) 05A15/CLC O517.1

Document code: A **Article ID**: 1000-341X(2001)01-0007-10

1. Introduction

We assume the terminology and notations of combinatorial analysis, most of them can be founded in [1] and [2]. For instance, $\binom{x}{k} = x(x-1)(x-2)\cdots(x-k+1)/k!$ denotes the k-order falling factor function which is a generalization of ordinary binomial coefficient and $\binom{n}{k}_q$ for its q-analogue, i.e., Gaussian binomial coefficient, $\binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$, where $(x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1})$. Let begin with a result of G.E.Andrew's [1] .

Theorem 1.1 *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of complex numbers such that*

$$\beta_n = \sum_{k=0}^{n} \frac{1}{(q;q)_{n-k}(aq;q)_{n+k}} \alpha_k$$

and further suppose that

$$\begin{cases} \alpha_n' = \dfrac{(\rho_1;q)_n(\rho_2;q)_n}{(aq/\rho_1;q)_n(aq/\rho_2;q)_n}\left(\dfrac{aq}{\rho_1\rho_2}\right)^n \alpha_n, \\[2ex] \beta_n' = \displaystyle\sum_{n\geq 0} \dfrac{(\rho_1;q)_k(\rho_2;q)_k(aq/\rho_1\rho_2;q)_{n-k}}{(q;q)_{n-k}(aq/\rho_1;q)_n(aq/\rho_2;q)_n}\left(\dfrac{aq}{\rho_1\rho_2}\right)^k \beta_k. \end{cases}$$

*Received date: 1998-04-22
Foundation item: Supported by the National Science Foundation of China (19771014)
Biography: MA Xin-rong (1964-), male, the Hui nationality, born in Pingluo county, Ningxia Hui nationality Autonomous Region, Ph.D., associate professor.
E-mail: xrma@public1.sz.js.cn

Then it holds

$$\beta_n' = \sum_{k=0}^{n} \frac{1}{(q;q)_{n-k}(aq;q)_{n+k}} \alpha_k'.$$

Andrews called this result Bailey chain and each pair α_n and β_n satisfying the above identity Bailey pair of inverse relation. Clearly, the procedure can start from (α_n, β_n) and only to produce (α_n', β_n') or conversely. Therefore, we may deduce an infinite family of Bailey pairs which is called Bailey chain from one initial pair. With such ideas G.E.Andrews [1] has found a simple but elegant proof of the famous Rogers-Ramamujan identities

$$\sum_{n\geq 0} \frac{q^{n^2}}{(q;q)_n} = \prod_{n=0}^{+\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})},$$

$$\sum_{n\geq 0} \frac{q^{n^2+n}}{(q;q)_n} = \prod_{n=0}^{+\infty} \frac{1}{(1-q^{5n+2})(1-q^{5n+3})}.$$

Such structure is of great practical value since it makes it possible for mathematician to find and check new mathematical results just like H.S.Wilf and D.Zeilberger have developed WZ-method. Furthermore, the Bailey chain are also related to q-hypergeometric series such as Jackson's formula, Waston's q-analogue of Whipple's theorem. In [4], Bressoud presented an extension of the Bailey chain which he called Bailey lattice.

The aim of the present paper is to give a sketch for such chain structure of arbitrary inverse relation and to investigate the inverse chain in great details for Gould-Hsu formula and its q-analogue. For this purpose, we need some basic concepts of matrix which have some slight differences with those in linear algebra. From now on, we assume that all matrix occur in the text are of infinite order, lower triangular and nonsingular.

Definition 1.1 *An infinite lower triangular matrix $A = (a_{n,k})$ over complex filed C is invertible or nonsingular if there exists a matrix $B = (b_{n,k})$ satisfying*

$$\sum_{k\leq j\leq n} a_{n,j}b_{j,k} = \sum_{k\leq j\leq n} b_{n,j}a_{j,k} = \delta_{n,k} \qquad \text{for all } n \geq k \geq 0.$$

Here, δ is the Kronecker delta function. It is clear that the above relation can also be stated as $AB = BA = E$ (infinite identity matrix). It is easy to see that infinite order lower triangular matrix $A = (a_{n,k})$ is nonsingular if and only if every entry $a_{k,k} \neq 0, k \in N$. For any two infinite complex sequences $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_n, \cdots)^T$ and $\beta = (\beta_0, \beta_1, \cdots, \beta_n, \cdots)^T$, $\alpha = A\beta$ is equivalent to $\beta = A^{-1}\alpha$ provided th at A is nonsingular. In terms of linear algebra, we restate this relation in the following forms:

$$\alpha_n = \sum_{k=0}^{n} a_{n,k}\beta_k \text{ if and only if } \beta_n = \sum_{k=0}^{n} b_{n,k}\alpha_k.$$

Remark It seems to be more precisely to say that the inverse relation mentioned here is just linear since that in the literature of combinatorial analysis there are yet nonlinear inverse relation, for example, the representative relations between the set of elementary symmetric functions and of power sum functions, because both sets are bases of space of symmetric functions.

Definition 1.2 *Suppose that there exists an inverse relation*

$$\alpha = A\beta,\ \text{that is } \beta = A^{-1}\alpha.$$

Let $\alpha' = D\alpha$ and $\beta' = C\beta$. If it still holds that $\beta' = A\alpha'$, then we call such procedure an inverse chain determinated by A and D and α and β is an inverse pair of inverse chain, where A^{-1} stands for the inverse matrix of A.

Following Andrew's notation, we write (A, D) for the inverse chain determinated by A and D and (α_n, β_n) for its inverse pair (α, β) in short. Evidently, the essence of the inverse chain can be displayed by the following commutative digram, where $\alpha \overset{A}{\mapsto} \beta$ means that $A\alpha = \beta$.

$$\begin{array}{ccc} \alpha & \xrightarrow{D} & \alpha' \\ A\downarrow & & \downarrow A \\ \beta & \underset{C}{\longrightarrow} & \beta' \end{array}$$

Figure 1.1

Theorem 1.2 *Given two arbitrary infinite matrixes A and D, where A is nonsingular. Then the inverse chain (A, D) exists if and only if $C = ADA^{-1}$, i.e., Figure 1.1 is commutative.*

Indeed, Theorem 1.2 is very general and it does not give too much information about the inverse chain of an inverse relation, but it provides a way to classify different combinatorial identities according to chain structure of inverse relation of inverse relation. For this purpose, we introduce the following concept.

Definition 1.3 *Two infinite order matrix A and B is defined to be equivalent if and only if there exist two nonsingular matrix M and N such that $A = NBM$, i.e., $B = N^{-1}AM^{-1}$. We write $A \approx B$ for the equivalent relation of A and B.*

Here are fundamental results about the inverse chain of inverse relation associated with the equivalent relation $\approx$. Note that for a given inverse chain (A, D), the set of inverse pairs of (A, D) forms a vector space with usual linear operations.

Theorem 1.3 *Suppose $A \approx B$ and there exists the inverse chain $(A; D)$. Let $P(A, D)$ be the vector space of inverse pairs of (A, D). Then*

(i) there exists the inverse chain $(B; MDM^{-1})$ where M and N are nonsingular matrixes such that $A = NBM$;

(ii) there exists an isomorphism ϕ from $P(A, D)$ of $(A; D)$ to $P(B, MDM^{-1})$ of $(B; MDM^{-1})$, such that

$$\phi(\alpha, \beta) = (M\alpha, N^{-1}\beta).$$

Proof It suffices to show this by straightforward verification, which are displayed in the following digraph.

$$\begin{array}{ccc} \alpha & \xrightarrow{M} & \sigma \\ A\downarrow & & \downarrow B \\ \beta & \underset{N^{-1}}{\longrightarrow} & \tau \end{array} + \begin{array}{ccc} \alpha & \xrightarrow{D} & \alpha' \\ A\downarrow & & \downarrow A \\ \beta & \underset{ADA^{-1}}{\longrightarrow} & \beta' \end{array} = \begin{array}{ccc} \sigma & \xrightarrow{MDM^{-1}} & \sigma' \\ B\downarrow & & \downarrow B \\ \tau & \underset{BMDM^{-1}B^{-1}}{\longrightarrow} & \tau' \end{array} \qquad \square$$

As mentioned earlier, this kind of equivalent relation can be used to classify the set of combinatorial identities which is a long-standing problem in combinatorics, according to inverse chains. It is reasonable to think so because such relation has been proven by I.Gessel[5] that the Bailey chain deduced from q-analogue of Lagrange inverse is of value to set up q-series identity. In addition, we restate the classical Bailey lemma in terminology of ours.

Theorem 1.4 *If* $\alpha = A\beta, \gamma = A^T\sigma$, *then* $\beta^T\gamma = \alpha^T\sigma$.

2. Inverse chains of Gould-Hsu formula

In this section, we will apply all the above observations to investigate the inverse chains of Gould-Hsu inverse formula as well as that of its q-analogue. One of the reasons for doing so is that Gould-Hsu formula is very fundamental and important in combinatorial analysis, so does its q-analogue in q-hypergeometric series. Furthermore although in the last section we shall give some main ideas of inverse chain, there are still a lot of works to do in order to make it more concrete and more rich, so we chose Gould-Hsu formulas as a representative illustration. Perhaps, the meaning of doing so is that we are able to find some new and deep relations for q-series and combinatorial identities. To establish the inverse chain of Gould-Hsu inverse, we first introduce the Gould-Hsu inverse [6].

Lemma 2.1 *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of complex numbers such that* $a_i + kb_i \neq 0$ *for all* $i, k = 0, 1, 2, \ldots.$, *and set*

$$\phi(x, n) = \prod_{i=0}^{n}(a_i + xb_i), \quad \phi(x, 0) \equiv 1.$$

Then

$$A = \left((-1)^k \binom{n}{k} \frac{a_{k+1} + kb_{k+1}}{\phi(n, k+1)}\right) \text{ equals that } A^{-1} = \left((-1)^k \binom{n}{k} \phi(k, n)\right).$$

Following the argument stated in last part, we are now about to set up two concrete inverse chains of Gould-Hsu formula under the situations for I. $\phi(x, n) = \frac{(n+y+x)!}{(x+y)!}$, which means that each $b_i = 1, a_i = i + y, i = 0, 1, 2, \cdots$, II. $\phi(x, n) \equiv 1$.

For both cases, we will use the following identity.

Lemma 2.2 *Let* $b \geq c \geq 0$ *be two real numbers. Then*

$$\sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{1}{\binom{b+k}{c}} = \frac{c}{(n+c)\binom{n+b}{b-c}}.$$

I. the first kind of Gould-Hsu inverse chain, is stated as below.

Theorem 2.1 *Let* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *be two sequences of complex numbers such that*

$$\beta_n = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{2k+1+y}{\binom{n+y+k+1}{n+y}} \frac{\alpha_k}{(k+1)!}$$

and

$$\begin{cases} \alpha_n' = \dfrac{\alpha_n}{2n+y+1}, \\ \beta_n' = \sum_{n\geq 0} \dfrac{\binom{n}{k}\binom{n+y}{k+y}}{(2(n-k)+1)\binom{2n+y+1}{2k+y}}\beta_k. \end{cases}$$

Then it holds

$$\beta_n' = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k+1+y}{\binom{n+y+k+1}{n+y}}\frac{\alpha_k'}{(k+1)!}.$$

Proof Based on Theorem 1.2, it suffices to calculate ADA^{-1}. Suppose that $ADA^{-1} = (d_{n,k})$ and $D = diag(\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n, \cdots)$ with $\lambda_n = \frac{1}{2n+1+y}$. Thus, we obtain that

$$d_{n,k} = \sum_{j=k}^{n} a_{n,j}\lambda_j a_{j,k}^{-1}.$$

By Lemma 2.1 and simple calculation, we deduce

$$d_{n,k} = \binom{n}{k}\sum_{j-k=0}^{n-k}(-1)^{j-k}\binom{n-k}{j-k}\frac{\binom{n+y}{k+y}}{(n-k+1)\binom{n+y+k+1+j-k}{n-k+1}}$$

$$\overset{\text{Lemma2.1}}{=\!=\!=\!=\!=\!=} \frac{\binom{n}{k}\binom{n+y}{k+y}}{(2(n-k)+1)\binom{2n+y+1}{2k+y}}.$$

Note that both paramers b and c in Lemma 2.1 are specified with $n+y+k+1$ and $n-k+1$, respectively. □

Note that $ADA^{-1} = (d_{n,k})$ if and only if $AD = (d_{n,k})A$. Thus it holds that $DA^{-1} = A^{-1}(d_{n,k})$. Thus, the Gould-Hsu inverse chain yields two basic combinatorial identities.

$$\sum_{j-k=0}^{n-k}\frac{\binom{n-k}{j-k}\binom{n+y}{j+k}}{(2(n-j)+1)\binom{2n+y+1}{2j+k}\binom{j+k+y+1}{j+y}} = \frac{1}{(2k+y+1)\binom{n+y+k+1}{n+y}},$$

$$\sum_{j-k=0}^{n-k}(-1)^{j-k}\frac{\binom{n-k}{j-k}\binom{n+y+j}{j-k}}{(2(j-k)+1)\binom{2j+1+y}{2k+y}} = \frac{1}{2n+y+1}.$$

Note that these two identities are not included in [7]. This fact displays again that the studies of Gould-Hsu inverse chain indeed can produce new relations.

II. the second kind of Gould-Hsu inverse chain or binomial-type of Gould-Hsu inverse chain. In [10], we have discussed this kind of inverse chain of Riordan group.

Theorem 2.2 *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of complex numbers such that*

$$\beta_n = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha_k$$

and

$$\begin{cases} \alpha_n' = \dfrac{1}{\binom{b+n}{c}}\alpha_n & b \geq c \geq 0, \\[2ex] \beta_n' = \displaystyle\sum_{n\geq 0} \dfrac{c\binom{n}{k}}{(c+n-k)\binom{n+b}{b-c+k}}\beta_k. \end{cases}$$

Then it holds

$$\beta_n' = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\alpha_k'.$$

The proof of this theorem is similar to that of Theorem 2.1 and is omitted here.

3. Applications to combinatorial sums and hypergeometric series

Now we will associate some particular pairs of Gould-Hsu inverse chain with two kinds of Gould-Hsu inverse chain to set up some useful transformations and relation of combinatorial identities. To do this, we often write the two kinds of Gould-Hsu chain in forms of combinatorial identities.

Theorem 3.1 *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be given by Theorem 2.1 and 2.2, respectively. Then*

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+y}{k+y}}{(2(n-k)+1)\binom{2n+y+1}{2k+y}}\beta_k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\alpha_k}{(k+1)!\binom{n+y+k+1}{n+y}}, \tag{1}$$

$$\sum_{k=0}^{n}\frac{c\binom{n}{k}}{(c+n-k)\binom{n+b}{b-c+k}}\beta_k = \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{1}{\binom{b+k}{c}}\alpha_k. \tag{2}$$

To display what these two inverse chain can produce, we looked into some basic identities listed in the book of Gould [7].

1.1. From identity (4.17) one can read out a pair of Gould-Hsu inverse chain

$$(\alpha_n,\beta_n) = \left(\frac{(n+1)!}{(2n+1)^2}, \frac{1}{\binom{n+1/2}{n}^2}\right) \qquad \text{while } y=0.$$

Then (1) leads to combinatorial identity

$$\sum_{k=0}^{n}\frac{\binom{n}{k}^2}{(2(n-k)+1)\binom{2n+1}{k}\binom{k+1/2}{k}^2} = \sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{(2k+1)^2\binom{n+k+1}{n}}.$$

1.2. The following identity is attributed to L.C.Hsu [8].

$$\sum_{k=0}^{n}\frac{\binom{n}{k}}{\binom{x+n+k}{n+1}} = \frac{(n+1)!2^n}{x(x+2)(x+4)\cdots(x+2n)},$$

it contains an inverse pair of Gould-Hsu inverse chain

$$(\alpha_n,\beta_n)=\left(\frac{(-1)^n(x+n-1)!}{x+2n},\frac{2^n(x+n-1)!}{(n+1)!x(x+2)\cdots(x+2n)}\right).$$

Taking it into (1), we can obtain that

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+y}{k+y}}{(2(n-k)+1)\binom{2n+y+1}{2k+y}}\frac{2^k(x+k-1)!}{(k+1)!x(x+2)\cdots(x+2k)}\\&=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{(x+k-1)!}{(x+2k)(k+1)!\binom{n+y+k+1}{n+y}}\qquad\text{while } y=x-1.\end{aligned}\tag{3}$$

In particular, let $x=2$ in (3), we find that

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{n+1}{k+1}}{(2(n-k)+1)(k+1)!\binom{2n+2}{2k+1}}=\frac{1}{n+1}.$$

1.3. From identity (4.20) in [7], we also read out an inverse pair

$$(\alpha_n,\beta_n)=\left(\frac{n^{2p}}{2n},0\right)\qquad\text{while}\quad y=-1, 1<2p<2n-1, p\ \text{ being integral.}$$

Then it holds

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{k^{2p-1}}{(k+1)!\binom{n+k}{n-1}}\equiv 0,\qquad 1<2p<2n-1, p\ \text{ being integral.}$$

The rest are some inverse pairs of binomial-type of Gould-Hsu inverse.

2.1. Let $b=c>0$. Obverse that there exists an inverse pair

$$(\alpha_n,\beta_n)=\left((-1)^n\binom{z}{n},\binom{z+n}{n}\right).$$

So (2) reduces to the following Vandermonde$'$ convolution identity

$$\sum_{k=0}^{n}\binom{b-1+n-k}{n-k}\binom{z+k}{k}=\binom{b+z+n}{n}.$$

2.2. From (7.6) in [7] it follows an inverse chain as

$$(\alpha_n,\beta_n)=\left(\binom{2n}{n}2^{-2n},\binom{n}{k}2^{-2n}\right)\qquad\text{while}\quad b=c>0.$$

Then (2) becomes to

$$\sum_{k=0}^{n}\frac{\binom{n}{k}\binom{2k}{k}}{2^{2k}\binom{n+b-1}{k}}=\frac{(b+n)\binom{2b+2n-1}{n+b}}{b2^{2n}\binom{2b-1}{b}}.$$

Further, let $b = 1$, then this result yields to

$$\sum_{k=0}^{n} 2^{-2k}\binom{2k}{k} = (2n+1)2^{-2n}\binom{2n}{n}.$$

It is a very interesting identity and can be checked directly.

2.3. Without any doubt, we also can decide one of identity once the rest are defined in Gould-Hsu inverse chain. Here is an example. From (7.6) in [7], to consider the case $b = c > 0$, we find

$$(\alpha_n', \beta_n') = \left(\frac{2^{2n}}{(2n+1)\binom{2n}{n}}, \frac{2^{2n}\binom{n-1/2}{n}}{(2n+1)\binom{2n}{n}}\right)$$

and calculate $\beta_n = \frac{2^{2n}\binom{n-b-1/2}{n}}{(2n+1)\binom{2n}{n}}$. Thus, from Gould-Hsu inverse chain it follows

$$\sum_{k=0}^{n} \frac{2^{2k}\binom{n}{k}\binom{k-b-1/2}{k}}{(2k+1)\binom{n+b-1}{k}} = \frac{(1+n)2^{2n}\binom{n-1/2}{n}}{b(2n+1)\binom{2n+1}{n}}.$$

2.4. From (7.9) in [7], let $b = c > 0$, we obtain $\alpha_n' = \frac{2^{2n}}{\binom{2n}{n}}$. After some simple calculations, we have

$$\beta_n' = \lim_{x \mapsto 1^+} \frac{(-1)^n\binom{2x}{2n}}{x\binom{x}{n}} = \frac{1}{2n-1}, \qquad \beta_n = \frac{(-1)^n\binom{2b+2}{2n}}{\binom{b+1}{n}}.$$

Then (2) leads to

$$\sum_{k=0}^{n} \frac{(-1)^k\binom{n}{k}\binom{2b+2}{2k}}{\binom{n+b-1}{k}\binom{b+1}{k}} = \frac{n+b}{b(2n-1)}.$$

2.5. From (4.30) in [7] there exists an inverse chain of binomial-type Gould-Hsu inverse as follows.

$$(\alpha_n, \beta_n) = \left(\frac{1}{\binom{b+n}{b}}, \frac{b}{n+b}\right).$$

Taking $b = c$ and applying this inverse pair in (2) produces

$$\frac{b^2}{n+b}\sum_{k=0}^{n} \frac{\binom{n}{k}}{(k+b)\binom{n+b-1}{k}} = \sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{\binom{b+k}{b}^2},$$

this identity covers a specified case while $b = 1$

$$H_n := \sum_{k=1}^{n}\frac{1}{k} = \sum_{k=1}^{n}(-1)^{k-1}\binom{n}{k}\frac{1}{k}.$$

which is a result related to the Harmonic series H_n.

4. Inverse chain of q-analogue of Gould-Hsu formula

We now consider the problem of inverse chain of q-analogue of Gould-Hsu inverse. The q-analogue of Gould-Hsu inverse was first obtained by L.Carlitz and then generalized by Andrews to Bailey lemma in Theorem 1.1. Now, we obtain from Theorem 1.1

$$A = (a_{n,k}) = \left(\frac{1}{(q;\ q\ )_{n-k}(aq;\ q)_{n+k}}\right),$$

and

$$A^{-1} = (a^{-1}_{n,\ k}) = \left((-1)^{n-k}q^{\binom{n-k}{2}}(1-aq^{2\ n})\frac{(a\ q;\ q)_{n+k-1}}{(q;\ q)_{n-k}}\right),$$

$$D = (d_{n,k}) = \left(\frac{(\rho_1;\ q)_k(\rho_2;\ q)_k}{(aq/\rho_1;q)_k(aq/\rho_2;\ q)_k}(\frac{aq}{\rho_1\rho_2})^k\right),$$

$$ADA^{-1} = \left(\frac{(\rho_1;\ q)_k(\rho_2;\ q)_k(aq/\rho_1\rho_2;\ q)_{n-k}}{(q;q)_{n-k}(aq/\rho_1;\ q)_n(aq/\rho_2;\ q)_n}(\frac{aq}{\rho_1\rho_2})^k\right).$$

Collecting these observations, we can obtain a new summation theorem concerning q-series.

Theorem 4.1

$$\sum_{j=k}^{n}(-1)^{j-k}q^{\binom{j-k}{2}}(1-aq^{2j})\binom{n-k}{j-k}_q\frac{(a;q)_{j+k-1}}{(aq;q)_{n+j}}\times$$
$$\frac{(\rho_1q^k;q)_{j-k}(\rho_2q^k;q)_{j-k}}{(aq^{k+1}/\rho_1;q)_{j-k}(aq^{k+1}/\rho_2;q)_{j-k}}\frac{aq}{\rho_1\rho_2})^{j-k}$$
$$=\frac{(aq/\rho_1;\ q)_k(aq/\rho_2;q)_k(aq/\rho_1\rho_2;q)_{n-k}}{(aq/\rho_1;\ q)_n(aq/\rho_2;\ q)_n}.$$

It contains the following specified cases which can be verified directly.

$$\sum_{k=0}^{+\infty}\frac{a^kq^{k^2}}{(q;q)_k(a;q)_k}=\frac{1}{(aq;q)_\infty},\qquad \sum_{k=0}^{n}\binom{n}{k}_q\frac{a^kq^{k^2}}{(a;q)_k}=\frac{1}{(aq;q)_n},$$

and

$$\sum_{j=0}^{+\infty}(-1)^jq^{\binom{j}{2}}(1-aq^{2j})\frac{(a;q)_{j-1}(\rho_1q;q)_j(\rho_2;q)_j}{(q;q)_j(aq/\rho_1;q)_j(aq/\rho_2;q)_j}(\frac{aq}{\rho_1\rho_2})^j$$
$$=\frac{(aq/\rho_1\rho_2;q)_\infty}{(aq/\rho_1;\ q)_\infty(aq/\rho_2;\ q)_\infty}.$$

Furthermore, one can deduce from $AD = ADA^{-1}A$ the finite form of Hein's formula of hypergeometric series

$$\sum_{j=0}^{n-k}\binom{n-k}{j}_q\frac{(\rho_1q^k;q)_j(\rho_2q^k;q)_j(aq/\rho_1\rho_2;q)_{n-k-j}}{(aq;q)_{2k+j}}(\frac{aq}{\rho_1\rho_2})^j$$
$$=\frac{(aq/\rho_1;q)_n(aq/\rho_2;q)_n}{(aq;q)_{n+k}(aq/rho_1;q)_k(aq/rho_2;q)_k}.$$

References:

[1] ANDREWS G E. *q-series: their development and applications in Analysis, Number Theory, Combinatorics,Physics, and Computer Algebra. Regional Conferences Series in Math* [M]. American Math. Society, Providence, Rhodeisland, 1986, **66**

[2] ANDREWS G E. *The Theory of Partitions* [M]. Encyclopedia of Math and Applications, Addison-Wesley, 1976, **2**

[3] COMTET L. *Advanced combinatorics* [M]. D.Reidel Publishing Company, 1972.

[4] BRESSOUD D M. *the Bailey lattice: an introduction, from Ramanujan Revisited* [M]. G.F.Andrews et al eds, Academic Press, New York, 1988.

[5] GESSEL I and STANTON D. *the application of q-analogue of Lagrange inversion* [J]. Transactions of the American Math Society, **277**:173–199.

[6] GOULD H W and HSU L C. *Some new inverse relations* [J]. Duke Math J., **40**, 1973.

[7] GOULD H W. *Combinatorial Identities* [M]. Morgantown Printing and Building Co., 1972.

[8] HSU L C. *Selected topics on methods and examples in Mathematical Analysis* [M]. Hight Educational Publishing House, 1955.

[9] RIORDAN J. *Combinatorial Identities* [M]. New York, 1968.

[10] MA Xin-rong. *The inverse chain of Riordan group and its applications to combinatorial sums* [J]. J. Math.Res & Exposition, 1999, **19**(2): 445–451.

一些反演关系的反演链

马欣荣[1]，王天明[2]

(1. 苏州大学理学院数学系，江苏苏州 215006；2. 大连理工大学应用数学系，辽宁大连 116024)

摘 要: 本文主要提出和讨论了一般反演关系的反演链和反演对的理论. 在此基础上，具体给出 Gould-Hsu 反演的两类反演链和 Gould-Hsu 反演的 q- 模拟的相关问题. 最后，本文结合一些常用的反演对，建立了众多的组合恒等式.