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On Diagonalization of Idempotent Matrices over APT Rings *

GUO Xue-jun, *SONG Guang-tian*
(Dept. of Math., University of Science and Technology of China, Heifei 230026, China)

Abstract: Let R be an abelian ring (all idempotents of R lie in the center of R), and A be an idempotent matrix over R. The following statements are proved: (a). A is equivalent to a diagonal matrix if and only if A is similar to a diagonal matrix. (b). If R is an APT (abelian projectively trivial) ring , then A can be uniquely diagonalized as $\mathrm{diag}\{e_1, ..., e_n\}$ and e_i divides e_{i+1}. (c). R is an APT ring if and only if R/I is an APT ring, where I is a nilpotent ideal of R. By (a), we prove that a separative abelian regular ring is an APT ring.

Key words: Abelian ring; APT ring; idempotent matrix.

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1. Introduction

Whether an idempotent matrix over a ring R can be daigonalized under a similarity transformation is an interesting problem not only in matrix theory, but also in module theory and algebraic K-theory. In module theory, this problem becomes whether a finitely generated projective R-module can be written as a direct sum of Re_i, where e_i's are idempotents of R. And by Theorem 1.2.3 of [1], $K_0(R)$ can be computed from the idempotent matrices over R.

Let R be a commutative ring with identity. In 1945, Foster proved the following theorem for a commutative ring R with identity (see [2] for details): Each idempotent matrix over R is diagonalizable under a similarity transformation if and only if each idempotent matrix over R has a characteristic vector.

Recall that a ring with identity R is a PT (projectively trivial) ring if every idempotent matrix over R is similar to a diagonal matrix. In 1966, Steger in [3] utilized Foster's Theorem to prove the following two theorems:

Theorem A *Let R be a commutative ring with identity and A be an $n \times n$ idempotent*

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Biography: GUO Xue-jun (1973-), male, Ph.D.

matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix, then there is an invertible matrix U such that UAU^{-1} is a diagonal matrix.

Theorem B *Let R be a commutative PT ring and A be an $n \times n$ idempotent matrix over R. Then*

(a) There is an invertible matrix P with $PAP^{-1} = \mathrm{diag}\{a_1, a_2, ..., a_n\}$ where a_i divides a_{i+1} for $1 \le i \le n-1$.

(b) If Q is another invertible matrix with $QAQ^{-1} = \mathrm{diag}\{b_1, b_2, ..., b_n\}$ where b_i divides b_{i+1} for $1 \le i \le n-1$, then $b_i = a_i$ for $1 \le i \le n$.

We call R an abelian ring, if R is a ring with identity and all idempotents of R lie in the center of R. We will demonstrate in this paper that Foster's Theorem and Theorem A can be generalized to abelian rings (Theorem 4 and Theorem 1 separately). Since there are no usual trace and determinant functions for matrices over noncommutative rings, the methods in Foster's and Steger's proofs need to be improved. As an application, we generalize Theorem B to APT (abelian projectively trivial) rings (Theorem 5).

Let I be a nilpotent ideal of a ring R, we prove that R/I is an APT ring if and only if R is an APT ring (Theorem 8). Then we prove that $R[x_1, x_2, ..., x_n]$ is an APT ring if and only if $(R/I)[x_1, x_2, ..., x_k]$ is an APT ring.

In [4], the authors defined a ring with identity to be separative if for any finitely generated projective R-modules A and B, $2R \oplus A \simeq R \oplus B$ implies $R \oplus A \simeq B$. Finally, as an application of Theorem 1, we point out that a separative abelian regular ring is an APT ring (Theorem 10).

2. Main results

Theorem 1 *Let R be an abelian ring and A be an $n \times n$ idempotent matrix over R. If there exist invertible matrices P and Q such that PAQ is a diagonal matrix, then there is an invertible matrix U such that UAU^{-1} is a diagonal matrix.*

Proof Suppose that there exist invertible matrices P and Q such that $PAQ = \mathrm{diag}\{b_1, b_2, \cdots, b_n\} = B$. Set $U = Q^{-1}P^{-1} = (u_{ij})$, then $(BU)^2 = BU$ and $BUB = B$, which implies $b_i = b_i u_{ii} b_i$, and so, $b_i u_{ii}$ and $u_{ii} b_i$ are idempotents of R. Set $e = b_i u_{ii}$, then $b_i = e b_i$ and $b_i = e(1-e+b_i)$. It is easy to verify that $(1-e+b_i)(1-e+eu_{ii}) = 1 = (1-e+eu_{ii})(1-e+b_i)$ So $1 - e + b_i$ is a unit, $b_i u_{ii}$ and b_i differ by a unit factor. Thus we may assume that Q has been adjusted so that $b_i^2 = b_i, i = 1, 2, ..., n$. The matrix equality $BUB = B$ gives (a) $b_i u_{ii} = b_i, i = 1, 2, ..., n$;

(b) $b_i b_j u_{ij} = 0, i \ne j; i, j = 1, 2, \cdots, n$.

Let D be an $n \times n$ matrix whose (i, j) entry is $b_i u_{ij}$ if $i \ne j$ and 1 if $i = j$, then $D^2 = 2D - I_n$, so D is invertible. It is easy to verify that $DBU = BD$. Thus, $(DP)A(DP)^{-1} = D(PAP^{-1})D^{-1} = D(PAQQ^{-1}P^{-1})D^{-1} = DBUD^{-1} = B$. □

Theorem 2 *Let R be an APT ring. Then any right (left) unimodular vector $(a_1, a_2, ..., a_n)$ in R^n is completable (i.e., it can be seen as the first row of some invertible matrix).*

Proof Suppose that $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is right unimodular. Let $\beta = (\beta_1, \beta_2, ..., \beta_n)^t \in R^n$ such that $\alpha_1\beta_1 + \alpha_2\beta_2 + \cdots + \alpha_n\beta_n = 1$. Set $A = \beta\alpha = (\beta_i\alpha_j)$, then $A^2 = A$. Since R is an APT ring, there exists an invertible matrix P with $PAP^{-1} = B = \text{diag}\{e_1, e_2, ..., e_n\}$. Let $X = (x_1, ..., x_n) = \alpha P^{-1}$, $Y = (y_1, ..., y_n)^t = P\beta$. Then $XY = \alpha\beta = 1$, $YX = PAP^{-1} = \text{diag}\{e_1, ..., e_n\}$, and α is completable iff X is completable. Since $\sum_{i=1}^n x_i y_i = 1$, $y_i x_i = e_i$, and $y_i x_j = 0$ $(i \neq j)$, so $y_i x_i y_i = y_i$, $x_i y_i x_i = x_i$. Thus $e_i = y_i x_i$ and $f_i = x_i y_i$ are idempotents. Since R is an abelian ring, e_i and f_i are in the center of R, $e_i = e_i^2 = y_i x_i y_i x_i = y_i f_i x_i = f_i e_i$, $f_i = f_i^2 = x_i y_i x_i y_i = x_i e_i y_i = f_i e_i$, so $e_i = f_i$, i.e., $x_i y_i = y_i x_i$. Thus $(\sum_{i=1}^n x_i)(\sum_{i=1}^n y_i) = (\sum_{i=1}^n y_i)(\sum_{i=1}^n x_i) = \sum_{i=1}^n x_i y_i = 1$, this means $x = \sum_{i=1}^n x_i$ is a unit of R. Let

$$D = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ -1 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \cdots & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{pmatrix}.$$

Then

$$DP = \begin{pmatrix} x & x_2 & \cdots & x_n \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

is invertible, so D is invertible, and hence X is completable. This implies that α is completable.

Corollary 3 *If R is an APT ring, then any stably free R-module is free.*

Proof It follows from Theorem 2 and Corollary 4.5 of [5]. □

Let R be a ring, A be an idempotent matrix over R and $\alpha \in R^n$ be a unimodular vector. Recall that α is a characteristic vector of A if α is completable and $A\alpha = \alpha\lambda$ for some λ in R.

The following theorem generalizes Foster's Theorem (see [2]) to abelian rings.

Theorem 4 *The following statements are equivalent for an abelian ring R:*

(a) Each idempotent matrix over R is diagonalizable under a similarity transformation.

(b) Each idempotent matrix over R has a charateristic vector.

Proof Suppose that we have (a) and A is an $n \times n$ idempotent matrix, then there is an invertible matrix Q with $QAQ^{-1} = \text{diag}\{e_1, ..., e_n\}$. Let $\alpha = (1, 0, ..., 0)^t$, then α is completable. Further, $QAQ^{-1}\alpha = e_1\alpha$. Set $\beta = Q^{-1}\alpha$, then β is completable, and $A\beta = e_i\beta$. Hence A has a characteristic vector.

Suppose that we have (b) and A is an $n \times n$ idempotent matrix. Assume that (a) is true for all idempotent matrices of size $< n$. If $A = 0$, there is nothing to prove. Assume that $A \neq 0$. Let α be a characteristic vector of A and $A\alpha = e_1\alpha$. Set $\beta_1 = \alpha$. Let $\beta_1, \beta_2, ..., \beta_n$ be a basis of R^n. Employing the basis $\beta_1, \beta_2, ..., \beta_n$ of R^n, the matrix A has the form

$$A_1 = \begin{pmatrix} e_1 & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \quad B = \begin{pmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

Since $A_1^2 = A_1$, we have $e_1^2 = e_1$, then $B^2 = B$. By the induction hypothesis, the matrix B may be diagonalized under a suitable similarity transformation. Thus by a suitable change of the basis, we may assume that we have chosen a new basis $\alpha_1, \alpha_2, ..., \alpha_n$ of R^n such that, relating to this basis, A has the form

$$A_2 = \begin{pmatrix} e_1 & b_2 & b_3 & \cdots & b_n \\ 0 & e_2 & 0 & \cdots & 0 \\ 0 & 0 & e_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & e_n \end{pmatrix}.$$

Since $A_2^2 = A_2$, we have $e_1^2 = e_1$, ..., $e_n^2 = e_n$, and $b_2(e_1 + e_2 - 1) = 0, ..., b_n(e_1 + e_n - 1) = 0$. Let $P = I_n + E_{12} + \cdots + E_{1n}$, where E_{ij} is an $n \times n$ matrix whose only nonzero entry euqals to 1 and is in (i, j) position. Where $r_i = b_i(1 - 2e_i)$, then $PA_2P^{-1} = \text{diag}\{e_1, e_2, ..., e_n\}$. □

The next theorem generalizes Theorem B to APT rings.

Theorem 5 *Let R be an APT ring and A be an $n \times n$ idempotent matrix over R. Then*

(a) There is an invertible matrix P with $PAP^{-1} = \text{diag}\{a_1, a_2, ..., a_n\}$ where a_i divides a_{i+1} for $1 \le i \le n-1$.

(b) If Q is another invertible matrix with $QAQ^{-1} = \text{diag}\{b_1, b_2, ..., b_n\}$ where b_i divides b_{i+1} for $1 \le i \le n-1$, then $b_i = a_i$ for $1 \le i \le n$.

Proof Suppose that P is an invertible matrix with $PAP^{-1} = \text{diag}\{e_1, ..., e_n\}$, Let $a_1 = 1 - (1-e_1)(1-e_2)\cdots(1-e_n)$ and $x_i = e_i + (1-e_1)(1-e_2)\cdots(1-e_n)$, then $a_1 x_i = e_i$ and $I(x_1, ..., x_n) = R$, i.e., $x_1, ..., x_n$ generate R. By Theorem 2, $X = (x_1, ..., x_n)$ is completable, so X is a characteristic vector of A. Then, in a fashion analogous to the proof of Theorem 4, we have that A is similar to $\text{diag}\{a_1, e_2', ..., e_n'\}$, by induction on $n-1$ (the size of the matrix). Assume that A is similar to $\text{diag}\{a_1, a_2, ..., a_n\}$ where a_i divides a_{i+1} for $2 \le i \le n-1$. Since a_1 divides each entry of $\text{diag}\{e_1, ..., e_n\}$, and $\text{diag}\{a_1, a_2, ..., a_n\}$ is similar to $\text{diag}\{e_1, ..., e_n\}$, we have that a_1 divides a_2. This completes part (a).

To show (b), observe that a_r divides the products of arbitrary r entries of $\text{diag}\{a_1, a_2, ..., a_n$ so a_r divides the product of any r entries of $\text{diag}\{b_1, b_2, ..., b_n\}$. Since b_i is idempotent and $b_i | b_{i+1}$, $b_r = b_1 b_2 \cdots b_r$, so $a_r | b_r$. Similarly, $b_r | a_r$. Since a_r and b_r are idempotents, we have $a_r = b_r$, $1 \le r \le n$. □

Proposition 6 *Let R be an APT ring, and P be a nonzero finitely generated projective R-module.*

(a) There exist nonzero idempotents $e_1, e_2, ..., e_n$ of R such that e_i divides e_{i+1} for $1 \le i \le n-1$ and $P \simeq Re_1 \oplus Re_2 \oplus \cdots \oplus Re_n$.

(b) If there are other nonzero idempotents $f_1, f_2, ..., f_m$ of R such that f_i divides f_{i+1} for $1 \le i \le m-1$ and $P \simeq Rf_1 \oplus Rf_2 \oplus \cdots \oplus Rf_m$, then $m = n$ and $e_i = f_i$ for $1 \le i \le n$.

Proof For an arbitrary projective R-module P, there is an idempotent matrix $A \in M_s(R)$ such that P is isomorphic to the image of the R-linear mapping $f : R^s \longrightarrow R^s$ which is defined by $f(\alpha) = A\alpha$ for any column vector α in R^s. By the above theorem there exists a basis $\alpha_1, \alpha_2, ..., \alpha_s$ of R^s such that $A\alpha_i = e_i\alpha_i$ for idempotent element e_i such that e_i divides e_{i+1} for $1 \le i \le s-1$. So P is isomorphic to $Re_1 \oplus Re_2 \oplus \cdots \oplus Re_s$ which is the image of f. Assume that n is the largest index such that $e_n \ne 0$, then (a) is proved.

To show (b), Let $B = \mathrm{diag}\{f_1, f_2, ..., f_m\}$, then B is corresponding to P in the same fashion as the above, then by 1.2.1 of [1] B is 0-similar to $\mathrm{diag}\{e_1, e_2, ..., e_n\}$, which means there exists a surfficient large positive integer k such that

$$\mathrm{diag}(e_1,\ e_2,\ ...,\ e_n,\ 0_{k-n}) \simeq \mathrm{diag}(f_1,\ f_2,\ ...,\ f_m,\ 0_{k-m}).$$

By (b) of Theorem 5, $m = n$ and $e_i = f_i$ for $1 \le i \le n$. □

Corollary 7 *If R is an APT ring, then $R^m \simeq R^m \oplus K$ implies $K = 0$.*

Proof It follows from Proposition 6. □

Theorem 8 *Let R be an abelian ring, I be a nilpotent ideal of R. Then R/I is an APT ring if and only if R is an APT ring.*

Proof Suppose that R/I is an APT ring. Let $f : R \longrightarrow R/I;\ r \mapsto \overline{r} = f(r)$ be the natural morphism. The "bar" notation will also be used for all vectors $\in R^n$ and all $n \times n$ matrices in $M_n(R)$.

Suppose that A is an idempotent matrix in $M_n(R)$. Let $\overline{A} = f(A)$. Then $\overline{A}$ is idempotent in $M_n(R/I)$. So $\overline{A}$ is similar to $\mathrm{diag}\{\overline{a}_1, \overline{a}_2, ..., \overline{a}_n\}$, where $\overline{a}_i$ divides $\overline{a}_{i+1}$. Since I is a nilpotent ideal, by 27.1 of [6], all the idempotents in R/I can be lifted modulo I. So there is an idempotent d in R such that $f(d) = \overline{a}_1$. By Theorem 4, $\overline{A}$ has a charateristic vector $\overline{x} = (\overline{x}_1, \overline{x}_2, ..., \overline{x}_n)^t$ corresponding to $\overline{a}_1 = \overline{d}$. Let x_i be in R with $f(x_i) = \overline{x}_i$, $1 \le i \le n$. Set $x = (x_1, x_2, ..., x_n)^t$, then since $\overline{x}$ is completable to $\overline{X}$ in $\mathrm{GL}_n(R/I)$ and $f : \mathrm{GL}_n(R) \longrightarrow \mathrm{GL}_n(R/I)$ is surjective, x is unimodular and completable. Then $Ax = dx + r$, where $r = (r_1, r_2, ..., r_n)^t$ with $r_i \in I$. Since $A^2 = A$ and $d^2 = d$, $Ax = dAx + Ar$ and $(1-d)d = 0$, $Ar = (1-d)Ax = (1-d)(dx + r) = (1-d)r$. Thus

$$\begin{aligned} A(x + (2d-1)r) &= Ax + (2d-1)Ar = dx + r + (2d-1)(1-d)r \\ &= dx + dr = d(x + (2d-1)r). \end{aligned}$$

Further, $x + (2d-1)r \equiv x(\mathrm{mod} I)$. Hence, as the above, $x + (2d-1)r$ is unimodular and completable. Thus A has a characteristic vector. By Theorem 4, R is an APT ring.

Assume that R is an APT ring and $\overline{A} = (\overline{A})^2 = (\overline{a}_{ij}) \in M_n(R/I)$. It is sufficient to prove that there exists an idempotent matrix $F = (f_{ij}) \in M_n(R)$ such that $\overline{F} = \overline{A}$. If $A = (a_{ij})$, then $A^2 = A + B$, where the entries of B are in I. Thus B is a nilpotent matrix. Let k be the least natural number such that $B^k = 0$. If $k = 1$, there is nothing left to prove. Assume that $k > 1$ and let $C = A + (I - 2A)B$. Then the entries of $C - A$ are in

I, and since $AB = BA = A^3 - A^2$, $C^2 = A^2 + 2A(I-2A)B + (I-2A)^2B^2$. Therefore, $C^2 - C = B + (I-2A)^2(B^2 - B)$. Since $(I-2A)^2 = I + 4B$, $C^2 = C + B^2(4B - 3I)$. Let $D = B^2(4B - 3I)$, then $C^2 = C + D$ where the entries of D are in I, and for some natural integer $l < k$, $D^l = 0$. Repeating this process, we arrive in a finite number of steps at the required matrix F. □

Corollary 9 *Let I be a nilpotent ideal of an APT ring R and let $x_1, x_2, ..., x_k$ be indeterminates. Then $R[x_1, x_2, ..., x_n]$ is an APT ring if and only if $(R/I)[x_1, x_2, ..., x_k]$ is an APT ring.*

Proof The corollary follows by observing that $I[x_1, x_2, ..., x_k]$ is a nilpotent ideal in $R[x_1, x_2, ..., x_k]$ and that $R[x_1, x_2, ..., x_k]/I[x_1, x_2, ..., x_k] \simeq (R/I)[x_1, x_2, ..., x_k]$. □

Theorem 10 *If R is a separative abelian regular ring, then R is an APT ring.*

Proof By Theorem 2.5 in [4], every square matrix over R admits a diagonal reduction (i.e., there exist invertible matrix P and Q such that PAQ is a diagonal matrix). Suppose A is an idempotent matrix, by Theorem 1, A is similar to a diagonal matrix whose diagonal entries are idempotents of R. So R is an APT ring. □

References:

[1] ROSENBERG J. *Algebraic K-Theory and Its Applications* [C]. Gradudate Texts in Mathematics 147, Springer-Verlag, 1995.

[2] MCDONALD B R. *Linear Agebra over Commutative Rings* [M]. Marcel Dekker, 1984.

[3] STEGER A. *Diagonability of idempotent matrices* [J]. Pac. J. Math., 1966, **19**(3): 535–541.

[4] ARA P, GOODEARL K R, O'MEARA K C et. al. *Diagonalization of matrices over regular rings* [J]. Lin.Alg.Appl. 1997, **265**: 136–147.

[5] LAM T Y. *Serre's Conjecture* [M]. Lecture Notes in Mathematics 635, Springer-Verlag, 1978.

[6] ANDERSON F And FULLER K. *Rings and Categories of Modules* [M]. Springer-Verlag, 1974.

APT 环上幂等阵的对角化

郭 学 军， 宋 光 天

(中国科学技术大学数学系，安徽合肥 230026)

摘 要: 设 R 是一阿贝尔环 (R 的所有幂等元都在中心里),A 是 R 上的一幂等阵. 本文证明了以下结果： (a) A 相抵于一对角阵当且仅当 A 相似于一对角阵； (b) 若 R 是一 APT(阿贝尔投射平凡) 环，则 A 在相似变换之下可唯一地化为对角形 diag$\{e_1, \cdots, e_n\}$, 这里 e_i 整除 e_{i+1}; (c) R 是 APT 环当且仅当 R/I 是 APT 环，这里 I 是环 R 的一幂零理想. 由 (a), 还证明了分离的阿贝尔正则环是 APT 环.