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A Characteristic of Dedekind Groups *

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Abstract: In this paper, we show that if a finite group G has a fixed-point-free weak power automorphism, then G is a Dedekind group.

Key words: Dedekind Group; fixed-point-free weak power automorphism; nonnormal cyclic subgroup.

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[1] Lemma 3 proved the following result: A finite group G of odd order is abelian if and only if G has a fixed-point-free power automorphism. In this paper, we introduce a weaker concept than that of power automorphism and get a new characteristic of Dedekind groups. The argument is quite different from [1]. The terms and notation are referred to [2].

Definition *Let α be an automorphism of a group G. Then α is called a power automorphism if for every subgroup H of G, $H^\alpha \leq H$; α is called a weak power automorphism if for every nonnormal subgroup H of G, $H^\alpha \leq H$; α is called a fixed-point-free automorphism if the identity of G is the unique fix point of α.*

Lemma 1 *Let G be a finite p–group($p > 2$) with a weak power automorphism α. If G contains a subgroup N of order p such that $N^\alpha \neq N$, then G is abelian.*

Proof Ob viously, $N \leq Z(G)$ and $N^\alpha \leq Z(G)$. Let U be a nonnormal cyclic subgroup of G. Then $U \cap N = 1$. Let $H = UN$. If $H^\alpha \neq H$, then H^α, H are normal in G. It follows that $H \cap H^\alpha = U$ is normal in G, a contradiction. Hence $H = H^\alpha$. Let V be another nonnormal cyclic subgroup. In the same way, $V \cap N = 1$, and VN is α-invariant. Hence $UN \cap VN = \langle c\rangle N$, $\langle c\rangle \neq 1$. So $\langle c\rangle N$ is α-invariant. It follows that $NN^\alpha \leq \langle c\rangle N$. Since any nonnormal subgroup contains a nonnormal cyclic subgroup, all nonnormal subgroups of G/N contain NN^α/N. So G/N is abelian by [3, Theorem 1]. It follows that $G' \leq N$,

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and $G' \leq N^\alpha$. So $G' = 1$. Hence G is abelian.

Lemma 2 *Assume $G = \langle a, b | a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle$, where $n \geq 2, p \neq 2$. Then G has no fixed-point-free weak power automorphism.*

Proof If G has a weak power automorphism α which is fixed-point-free, let $H = \langle a^{p^{n-1}}, b | [a,b] = a^{p^{n-1}} \in G' \leq Z(G), |G'| = p \rangle$. Then $H \cong Z_p \times Z_p$, and α induced a power automorphism on H. Let $a^\alpha = a^u b^v$ for some integers u and v, where $u \not\equiv 0$ (mod p)

$$[a,b]^\alpha = (a^{p^{n-1}})^\alpha = (a^\alpha)^{p^{n-1}} = (a^u b^v)^{p^{n-1}} = a^{up^{n-1}} b^{vp^{n-1}} [b,a]^{\frac{p^{n-1}(p^{n-1}-1)}{2}}$$
$$= a^{up^{n-1}} = (a^{p^{n-1}})^u = [a,b]^u.$$

On the other hand,

$$[a,b]^\alpha = [a,b]^u = [a^u b^v, b^u] = [a^u, b^u]^{b^v} [b^v, b^u] = [a^u, b^u] = [a,b]^{u^2}.$$

(here $G' \leq Z(G)$ is used), so $u \equiv u^2$ (mod p), i.e., $u \equiv 1$ (mod p). By [2,Th.13,4.3] $b^\alpha = b^u$. That implies $b^\alpha = b^u = b$, a contradiction. Therefore G has no fixed-point-free weak power automorphism.

Lemma 3 *Let $G = \langle a, b, c | a^p = b^p = c^p = [a,c] = [b,c] = 1, [a,b] = c \rangle$ and $p \neq 2$. Then G has no fixed-point-free weak power automorphism.*

Proof The proof is straightforward.

Theorem 1 *Let p be a prime and G be a finite $p-$group. If G has a fixed-point-free weak power automorphism, then G is a Dedekind group.*

Proof Case 1 $p \neq 2$. Let G be a counterexample of minimal possible order, and α be a fixed-point-free weak power automorphism. If α is a power automorphism, then Theorem 1 is true by [1, Lemma 3]. Assume there exists at least $N \lhd G$ such that $N^\alpha \neq N$. Let P be an subgroup of order p in $Z(G)$. By Lemma 1 we obtain $P^\alpha = P$. α induced a fixed-point-free weak power automorphism on G/P. By the minimality of G, G/P is abelian, i.e., $G' = P$. That implies $Z(G)$ is cyclic and G has a unique minimal normal subgroup. we obtain that $|G'| = p$ and $G' \leq Z(G)$.

If G contains a nonnormal cyclic maximal subgroup, then G is known by [2. 5.3.4]. It is easy to get by checking that G has no fixed-point-free power automorphism, a contradiction.

If G dosen't contain any cyclic maximal subgroup, we will prove G contains a α-invariant subgroup which is isomorphic to the group of Lemma 3.

First, note that if $x \in G$ and $< x >$ is not normal in G, then $|\langle x \rangle| = p$. In fact, let $H = \langle x \rangle G'$. Since $G' \leq Z(G)$, H is an abelian normal subgroup of G, and so $H^p = \langle x^p \rangle \lhd G$, hence $H^p \neq 1$ if $|x| > p$. Since G has a unique minimal normal subgroup, we obtain $G' \leq \langle x \rangle$, which means $\langle x \rangle \lhd G$, a contradiction.

Second, note that for any subgroup $H$of G, $|G/C_G(H)| \leq p$. In fact, let $G^p = \langle x^p | x \in G \rangle$. Since $\forall a, b \in G, [a,b] \in G' \leq Z(G)$, $[x^p, a] = [x,a]^p = 1$. It follows that $G^p \leq C_G(H)$ for all subgroups H of G.

We consider $C_G(N)$, where $N^\alpha \neq N$. As above argument, $C_G(N)$ is maximal in G, so $C_G(N)$ is not cyclic. It follows that there exists a subgroup S of order p in $C_G(N)$ which is not normal in G. Let $C = C_G(S)$. Then C is maximal and α-invariant subgroup containing N. By the minimality of G, C is abelian. Let $y \bar{\in} C$. Then $G =< C, y >$ and $C \cap C_G(y)$ is contained in $Z(G)$. Since $G = C_G(S)C_G(y)$, and $C_G(S)C_G(y)/C_G(S) \cong C_G(S)/C_G(S) \cap C_G(y)$, $|G : C_G(S) \cap C_G(y)| = p^2$. It follows from the nonabelity of G that $Z(G) = C_G(S) \cap C_G(y)$. Hence $|G : Z(G)| = p^2$. Any maximal subgroup M of G containing $Z(G)$ is of the form $Z(G) \times \langle g \rangle$. We chose $M \neq C$, then $g \bar{\in} C$, and $\langle g \rangle$ is nonnormal in G(since G has a unique minimal normal subgroup, and $G' \leq C$.). So $|g| = p$. Let $H = \langle g, S \rangle$. Then H is a group of order p^3 and exponent p. It is isomorphic to the group of Lemma 3, a contradiction. This shows the counterexaple does not exist if $p \neq 2$.

Case 2 $p = 2$. If there exists at least a nonnormal subgroup of order 2, then α has a fix point. Assume all subgroups of order 2 are normal in G. Let $N \lhd G$ and $|N| = 2$. Then $N \leq Z(G), N \neq N^\alpha$, and $N^\alpha \leq Z(G)$. Let S be a cyclic nonnormal subgroup of G and $S_1 \leq S, |S_1| = 2$. Then $N \cap S = 1$. Let $H = SN$. If $H^\alpha \neq H$, then H, and H^α would be normal. Hence $S = H \cap H^\alpha$ is also normal, a contradiction. Therefore $H^\alpha = H$. Consider $\Omega(H) = \{x \in G | x^2 = 1\}$, obviously, $\Omega(H)$ is the unique elementary abelian subgroup of order 4 in H, so $\Omega(H) = N \times S_1$. On the other hand, $\Omega(H)$ is characteristic in H, i.e., $\Omega(H)$ is α -invariant, it follows that α induces a power automorphism on $\Omega(H)$, which fixes S_1, a contradiction. So the counterexample dose not exist if $p = 2$, Hence Theorem 1 is proved.

Example Let $G = \langle a, b | a^{p^n} = b^p = 1, a^b = a^{1+p^{n-1}} \rangle n \geq 2$. Then $\alpha : a \longmapsto a^u b^v, b \longmapsto b, u \equiv 1 \pmod p$, $v \not\equiv 0 \pmod p$ is a weak power automorphism, but not a power automorphism.

Proof Obviously, $\langle a \rangle^\alpha \neq \langle a \rangle$, we need to show α fixes all nonnormal subgroups of G. Note that G has unique minimal normal subgroup, and $G' = [a, b] = \langle a^{p^{n-1}} \rangle$ is a subgroup of order p and $G' \leq Z(G)$. then every nonnormal subgroup of G is of order p. So it is contained in $\Omega(G) = \langle a^{p^{n-1}}, b \rangle \cong Z_p \times Z_p$. By hypothesis, $b^\alpha = b$, and $(a^{p^{n-1}})^\alpha = (a^u b^v)^{\alpha^{n-1}} = a^{up^{n-1}} b^{vp^{n-1}} [b^v, a^u]^{\frac{p^{n-1}(p^{n-1}-1)}{2}} = a^{up^{n-1}} = a^{p^{n-1}}$. So α fixes all subgroup of $\Omega(G)$.

Theorem 2 *Assume a finite group G has a fixed-point-free weak power automorphism, then G is a Dedekind group.*

Proof If G is of prime-power order, then Theorem 2 is true by Theorem 1.

If G is not of prime-power order. let G be a counterexample of minimal possible order and α be a automorphism required of G. If $\forall N \leq G$, $N^\alpha = N$, then G is an abelian group of odd order by [1, Lemma 3]. We assume N is the minimal subgroup of G such that $N^\alpha \neq N$, then N is a cyclic normal subgroup of prime-power order. Thus there exists $x \in N$ such that $\langle x \rangle^\alpha \neq \langle x \rangle$. The minimality of N implies $N = \langle x \rangle$, in the same way, $\langle x \rangle$ is of prime-power order.

Let $N < P \in Syl_p(G)$. Then $\langle x \rangle$ is a normal subgroup of G. If y is an element of G with order prime to p, then $\langle x \rangle$ is characteristic in $\langle x \rangle \langle y \rangle$. Hence $\langle x \rangle \langle y \rangle$ is not fixed by α, and in particular, $\langle x \rangle \langle y \rangle$ is normal in G. If the subgroup $\langle y \rangle$ is not normal in $\langle x \rangle \langle y \rangle$, then $\langle x \rangle$ and $\langle x^{-1} y x \rangle$ are fixed by α. It follows that α fixes also $\langle x, y \rangle = \langle x^{-1} y x \rangle$. This is a contradiction. Therefore $\langle y \rangle$ is normal in $\langle x, y \rangle$, and hence also in G. That is, every p'-subgroup of G is normal. Let $Q \in \mathrm{Syl}_q(G) (q \neq p)$. Then $Q \lhd G$, in particular, Q is $\alpha-$invariant. On the other hand, $P^\alpha = P$, and $N^\alpha \neq N$. By the minimality of G, P is a Dedekind group, $N \leq O_p(G) \neq 1$. Since $O_p(G)$ is α-invariant, α induces a fixed-point-free weak power automorphism on $G/O_p(G)$ and G/Q respectively. Again use the minimality of G, we obtain $G/O_p(G)$ and G/Q are Dedekind groups. It follows that $G \cong G/O_p(G) \cap Q \cong$ a subgroup of $G/O_p(G) \times G/Q$ is nilpotent, in other words, G is Dedekind group, a contradiction. Theorem 2 holds.

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Dedekind 群的一个刻画

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摘 要: 本文给出了 Dedekind 群的一个刻画. 即如果一个群 G 有一个无不动点的弱幂同构，则 G 是一个 Dedekind 群.