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A Degree Condition for the Circumference of a Graph *

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Abstract: We present a new condition on the degree sums of a graph that implies the existence of a long cycle. Our result is a generalization of the previous one of N.Dean and P.Fraiss.

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1. Introduction and preliminary lemmas

Throughout, G will be a 2-connected simple graph with degree sequence

$$d(v_1) \le d(v_2) \le \cdots \le d(v_n).$$

For $j,k \in \{1,2,\cdots,n\}$, if $j < k$, $v_jv_k \in E(G)$, $d(v_j) \le j$ and $d(v_k) \le k-1$, then we call (j,k) a key-pair of G. Let $P = x_1x_2\cdots x_k$ be a path of G, where $x_i \in V(G)$. We may fix for P an orientation and set $x_j^+ = x_{j+1}$ for $1 \le j \le k-1$ and $x_j^- = x_{j-1}$ for $2 \le j \le k$. Define $X^+ = \{x^+ \mid x \in X\}$ and $X^- = \{x^- \mid x \in X\}$ for each subset $X \subseteq V(P)$. The length of a longest cycle of G, or *circumference* of G, is denoted by $C(G)$. The main result of this note is the following theorem:

Theorem 1 *Suppose G satisfies the property that for any key-pair (j,k) and a given positive integer m, we have*

(1) $d(v_j) + d(v_k) \ge m+1$, *if* $j+k \ge n$;

(2) $d(v_j)+d(v_k) \ge \min(k,m)$, *if* $j+k < n$ *and* $d(v_{k+1}) \ge \frac{m}{2}+1$ *when* $d(v_j)+d(v_k) = k$.

Then $C(G) \ge \min(m,n)$.

By considering an example given in [2], we can see that Theorem 1 is a generalization of a similar result of N.Dean and P.Fraisse [2].

To prove Theorem 1 we need the following well known lemma.

Lemma 2 *Let* $P = x_1x_2\cdots x_k$ *be a path of 2-connected graph* G. *Then* $C(G) \ge \min(|P|, d_P(x_1)+d_P(x_k))$.

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2. Proof of Theorem 1

Assume that Theorem 1 is false, and let G be a maximal graph that satisfies the hypothesis but not the conclusion. So we have $C(G) < m$. Let $P = v_j \cdots v_k$ be a longest path of G, then $|P| \geq m$, the neighbors of v_j and v_k lie on P and $v_j v_k \notin E(G)$. We may assume that $j < k$. Among the family of longest path in G, we choose P so that k is maximum and among the longest path with k maximum, we choose P so that j is maximum. So we have

Fact 1 For a vertex v_i of P, if $v_i^+ v_j \in E(G)$ (or $v_i^- v_j \in E(G)$), then $i \leq j$ ($i \leq k$).

Fact 2 $d(v_j) \leq j$ and $d(v_k) \leq k-1$.

Fact 3 $j + k < n$ and $d(v_j) + d(v_k) \geq k$.

By Fact 2, we have that (j, k) is a key-pair. Let $P = v_j \cdots v_k = x_1 \cdots x_{m'}$, where $x_1 = v_j$ and $x_{m'} = v_k$. We have $|P| = m' \geq m$. Suppose $m' \geq m + 2$ (for $m' \leq m+1$, the proof is similar).

Set $N = N(x_1)^- \cup N(x_{m'})^+$. Then, by Facts 1 and 3 , we have, for $v_i \in V(G) \backslash N$, $i \geq k+1$ and $d(v_i) \geq \frac{m}{2} + 1$. We need to consider the following two cases:

Case 1 If $x_1 x_r \in E(G)$ (or $x_r x_{m'} \in E(G)$) $(3 \leq r \leq m'-3)$, then

$$N(x_{r-2}) \subseteq V(P) (\text{or } N(x_{r'+2}) \subseteq V(P)).$$

Case 2 If $x_1 x_r \in E(G)$ and $x_{r'} x_{m'} \in E(G)$, then (1) $x_{r-2} \in N$ or $x_{r'+2} \in N$ when $2 < r \leq r' \leq m'-1$. When $r - 2 > r'$ and $2 < r', r < m-1$, we also have (2) $x_{r'-1} \in N$ or $x_{r-2} \in N$, (3) $x_{r'-2} \in N$ or $x_{r-2} \in N$, (4) $x_{r'+2} \in N$ or $x_{r+1} \in N$ and (5) $x_{r'+2} \in N$ or $x_{r+2} \in N$.

Let $t = \max\{i \mid x_1 x_i \in E, 2 \leq i \leq m'\}$ and $t' = \max\{i \mid x_i x_{m'} \in E, 1 \leq i \leq m'-1\}$.

For $t \leq t'$, we can get a contradiction from the discussion in Case 2. Suppose $t > t'$. If $x_1 x_r, x_{r'} x_{m'} \in E$, $r > r'$ and each vertex of $\{x_{r'+1}, \cdots, x_{r-1}\}$ is not adjacent to x_1 or $x_{m'}$, then we call $P[x_{r'}, x_r]$ an intersect interval of P. We have

(a) There exists only one intersect interval on P.

(b) $\{x_2, \cdots, x_{t'}\} \subseteq N(x_1)$ and $\{x_t, \cdots, x_{m'-1}\} \subseteq N(x_{m'})$.

(c) $E(\{x_{t'+1}, \cdots, x_{t-1}\}, \{x_1, \cdots, x_{t'-1}\}) = \emptyset$ and $E(\{x_{t'+1}, \cdots, x_{t-1}\}, \{x_{t+1}, \cdots, x_{m'}\}) = \emptyset$

Assume now that $P[x_{r'}, x_r]$ is the unique intersect interval of P. We have $r' = t'$ and $r = t$. Put $P_0 = P[x_t, x_{m'}]^{-1} P[x_1, x_{t-2}]$, we have, for every $v_f \in N_{P_0}(x_{m'})^-$, $f \leq k$,

$$N(x_{t-2}) \backslash V(P_0) = \{x_{t-1}\} \text{ and } N_{P_0}(x_{t-2}) \subseteq \{x_{t'}, \cdots, x_{t-3}\}.$$

Let $x_{t-2} = v_i$. Then $v_i \notin N$ and $i \geq k+1$. If there exists $x_r \in N_{P_0}(x_{t-2})^+$, $x_r = v_f$ and $f > i$, then put $P_0' = P_0[x_{m'}, x_r^-] P_0[x_r, x_{t-2}]^{-1}$, P_0' has the same property as P_0. Take the step repeatedly and also write the final path as $P_0 = v_k \cdots v_i$, where $i \geq k+1$ and $N(v_i) \subseteq V(P_0) \cup \{x_{t-1}\}$. Then $d(v_k) + d(v_i) \geq m$ and ,by (a), (b) and (c), we have

$$N(v_i) \cup N(v_k) \cup \{v_i, v_k\} \subseteq \{x_{t'}, \cdots, x_{m'}\}$$

and

$$N(v_k) \cap N(v_i) \subseteq \{x_{t'}, x_t\}.$$

So

$$|\{x_{t'}, \cdots, x_{m'}\}| \geq |N(v_i) \cup N(v_k)| + 2 \geq |N(v_i)| + |N(v_k)| - 2 + 2 \geq d(v_i) + d(v_k) \geq m.$$

Hence the cycle $C = P[x_{t'}, x_{m'}]x_{t'}$ has length at least m, a contradiction to the assumption.

The result of the theorem follows. □

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图的周长的一个度条件

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摘 要: 本文用度和给出一个关于图的最长圈的存在性条件. 这一结果是 N.Dean 和 P.Fraisse 的相应结论的推广.