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A Definition and Some Properties of Group-Valued Measure *

YANG Yi-chuan
(Dept. of Appl. Math., Beijing Polytechnic University, Beijing 100022, China)

Abstract: We give a definition and some primary properties of group-valued measure with a second countable complete Boolean algebra domain, and with a subset of a first countable complete Abelian po-group codomain.

Key words: first countable completness; second countable completness; Abelian po-group; Boolean algebra.

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1. Definition and notations

Definition 1.1[1] *A po-set G is said to be first countable complete, if for any countable bounded subset A of G,* $\sup A \in G$.

Definition 1.2[1] *A po-set G is said to be second countable complete, if for any countable subset A of G,* $\sup A \in G$.

Definition 1.3[2,3] *An Abelian group $G(+,0,\leq)$ is said to be a po-group, if it satisfies*

(i) $(G,\leq)$ *is a po-set,*

(ii) $\forall a,b,c \in G, a \leq b \Longrightarrow a+c \leq b+c$.

Here, the definition of Boolean algebra is the same as that in Sankappanavar[4].

Throughout this paper, **N** denotes the set of natural numbers, A denotes a second countable complete Boolean algebra, and G denotes a first countable complete Abelian po-group. Without loss of generality, we assume that there exists an element g of G, such that $g > 0$, and we define P_g by

$$P_g = \{x : (x \in G) \& (0 \leq x < g)\}.$$

Definition 1.4 *A mapping $\mu : A \longrightarrow P_g$ is a group-valued measure if it satisfies*

$$\mu\Big(\bigvee_{n=1}^{\infty} a_n\Big) = \sum_{n=1}^{\infty} \mu(a_n),$$

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Biography: YANG Yichuan (1971-), male, M.Sc.

where $\{a_n\}_{n=1}^{\infty}$ is a disjoint sequence of A . At this time, we call (A,μ) a group-valued measure space.

Remark 1.1 Obviously, the definition of group-valued measure given here is a generalization of general measure [5, 6], lattice measure [7], and set-valued measure[8] .

2. Theorems and proofs

Theorem 2.1 *Every second countable complete Boolean algebra is a countable distributive lattice.*

Proof First, it is easy to check that,

$$\bigvee_{n=1}^{\infty} a_n \in A, \bigwedge_{n=1}^{\infty} a_n \in A$$

for $\{a_n\}_{n=1}^{\infty} \subseteq A$.

Secondly, we suppose $a \in A$, and $\{a_n\}_{n=1}^{\infty} \subseteq A$. Putting $u = \bigvee_{n=1}^{\infty}(a \wedge a_n)$, then

$$(a \bigwedge a_n) \bigvee a' \leq u \bigvee a', \tag{1}$$

$$a \bigwedge a_n \leq a_n \tag{2}$$

for any $n \in \mathbf{N}$,where a' is the complementary element of a.

By the distributive laws , it follows from (1) that

$(a \wedge a_n) \vee a' = (a \vee a') \wedge (a_n \vee a') = (a_n \vee a') \Longrightarrow a_n \leq a_n \vee a' \leq u \vee a'$, that is $\bigvee_{n=1}^{\infty} a_n \leq u \vee a'$, and so

$$a \bigwedge (\bigvee_{n=1}^{\infty} a_n) \leq a \bigwedge (u \bigvee a') = a \bigwedge u \leq u.$$

It follows from (2) that $u \leq \bigvee_{n=1}^{\infty} a_n, a \wedge a_n \leq a \Longrightarrow u \leq a$, and consequentely, $u \leq a \wedge (\bigvee_{n=1}^{\infty} a_n)$. Hence, $u = a \wedge (\bigvee_{n=1}^{\infty} a_n)$.

By the similar arguments, we have

$$a \bigvee (\bigwedge_{n=1}^{\infty} a_n) = \bigwedge_{n=1}^{\infty} (a \bigvee a_n),$$

$$(\bigvee_{n=1}^{\infty} a_n) \bigwedge (\bigvee_{n=1}^{\infty} b_n) = \bigvee\nolimits_{m,n \in N} (a_n \bigwedge b_m),$$

and

$$(\bigwedge_{n=1}^{\infty} a_n) \bigvee (\bigwedge_{n=1}^{\infty} b_n) = \bigwedge\nolimits_{m,n \in N} (a_n \bigvee b_m). \qquad \square$$

Theorem 2.2 *If μ is a group-valued measure, then*

1) $\mu(0) = 0$;

2) μ has finite additivity :If $a_1, a_2, \cdots, a_n$ are disjoint elements , then

$$\mu\Big(\bigvee_{m=1}^{n} a_m\Big) = \sum_{m=1}^{n} \mu(a_m),$$

3) μ is monotone : $\mu(a) \le \mu(b)$ for $a \le b$;

4) μ is continuous from below : If $a_n \in A, a_n \le a_{n+1}$ for all $n \in N$, then

$$\mu\Big(\bigvee_{n=1}^{\infty} (a_n)\Big) = \bigvee_{n=1}^{\infty} \mu(a_n);$$

5) μ is continuous from above : If $a_n \in A, (n = 1, 2, \ldots), a_n \ge a_{n+1}$ for all $n \in N$, and there exists $k \in N$ such that $\mu(a_k) < g$, then

$$\mu\Big(\bigwedge_{n=1}^{\infty} a_n\Big) = \bigwedge_{n=1}^{\infty} \mu(a_n);$$

6) If $a_n \in A, (n = 1, 2, \ldots)$, then

$$\mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{n=1}^{\infty} a_n\Big) \le \bigvee_{n=1}^{\infty}\bigwedge_{n=1}^{\infty} \mu(a_n);$$

7) If $a_n \in A, (n = 1, 2, \ldots)$, and there exists $k \in N$ such that $\mu(\bigvee_{n=k}^{\infty} a_n) < g$,then

$$\mu\Big(\bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty} a_m\Big) \ge \bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty} \mu(a_m);$$

8)

$$If \quad a_n \in A, \quad (n = 1, 2, \ldots), \bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty} a_m = \bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty} a_m,$$

and there exists $k \in N$ such that $\mu(\bigvee_{n=k}^{\infty} a_n) < g$, then

$$\mu\Big(\bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty} a_m\Big) = \bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty} \mu(a_m).$$

Proof We will only verify 5) , 6) , 7) and 8), and the other properties of the measure follow similarly.

5) Without loss of generality , suppose $\mu(a_1) < g$. Let $b_n = a_1 \bigwedge a'_n, (n = 1, 2, \cdots)$, then $\{b_n\}$ satisfies 4) . So

$$\mu\Big(\bigvee_{n=1}^{\infty} b_n\Big) = \bigvee_{n=1}^{\infty} \mu(b_n).$$

Furthmore, it is easy to verify that

$$\Big(\bigvee_{n=1}^{\infty} b_n\Big) \bigvee \Big(\bigwedge_{n=1}^{\infty} a_n\Big) = a_1, \quad \Big(\bigvee_{n=1}^{\infty} b_n\Big) \bigwedge \Big(\bigwedge_{n=1}^{\infty} a_n\Big) = 0.$$

By 2), we have

$$\mu(a_1)=\mu(b_n)+\mu(a_n)=\mu\Big(\bigvee_{n=1}^{\infty}b_n\Big)+\mu\Big(\bigwedge_{n=1}^{\infty}a_n\Big)\Longrightarrow$$

$$\lim_{n\to\infty}(\mu(b_n)+\mu(a_n))=\lim_{n\to\infty}\Big(\mu\Big(\bigvee_{n=1}^{\infty}b_n\Big)+\mu\Big(\bigwedge_{n=1}^{\infty}a_n\Big)\Big)=$$

$$\mu\Big(\bigvee_{n=1}^{\infty}b_n\Big)+\mu\Big(\bigwedge_{n=1}^{\infty}a_n\Big)=\bigvee_{n=1}^{\infty}\mu(b_n)+\mu\Big(\bigwedge_{n=1}^{\infty}a_n\Big),$$

and

$$\lim_{n\to\infty}(\mu(b_n)+\mu(a_n))=\lim_{n\to\infty}\Big(\bigvee_{i=1}^{n}\mu(b_i)+\bigwedge_{i=1}^{n}\mu(a_i)\Big)=\bigvee_{i=1}^{\infty}\mu(b_i)+\bigwedge_{i=1}^{\infty}\mu(a_i).$$

Hence,

$$\bigvee_{i=1}^{\infty}\mu(b_i)+\bigwedge_{i=1}^{\infty}\mu(a_i)=\bigvee_{n=1}^{\infty}\mu(b_n)+\mu\Big(\bigwedge_{n=1}^{\infty}a_n\Big).$$

It follows that $\mu(\bigwedge_{n=1}^{\infty}a_n)=\bigwedge_{n=1}^{\infty}\mu(a_n)$.

6) Let $b_k=\bigwedge_{n=k}^{\infty}a_n$, then $\{b_k\}$ satisfies 4) , and $b_k\le a_k$. Therefore,

$$\mu\Big(\bigvee_{k=1}^{\infty}b_k\Big)=\mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty}a_m\Big)=\bigvee_{k=1}^{\infty}\mu(b_k)=\bigvee_{k=1}^{\infty}\mu\Big(\bigwedge_{n=k}^{\infty}a_n\Big)\le\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}\mu(a_n).$$

7) Let $b_k=\bigvee_{n=k}^{\infty}a_n$, then $\{b_k\}$ satisfies 5), and $b_k\ge a_k$. So

$$\mu\Big(\bigwedge_{k=1}^{\infty}b_k\Big)=\mu\Big(\bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty}a_m\Big)=\bigwedge_{k=1}^{\infty}\mu(b_k)=\bigwedge_{k=1}^{\infty}\mu\Big(\bigvee_{n=k}^{\infty}a_n\Big)\ge\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty}\mu(a_n).$$

8) By 6) and 7),

$$\mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty}a_m\Big)\le\bigvee_{k=1}^{\infty}\bigwedge_{n=k}^{\infty}\mu(a_n)\le\bigwedge_{k=1}^{\infty}\bigvee_{n=k}^{\infty}\mu(a_n)$$
$$\le\mu\Big(\bigwedge_{n=1}^{\infty}\bigvee_{m=n}^{\infty}a_m\Big)=\mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty}a_m\Big).\qquad\square$$

Theorem 2.3 *Let (A,μ) be a group-valued measure space , $n\in A$. n is negligible (if there exists $a\in A$, such that $\mu(a)=0$ and $n\le a$). If $W=\{n\in A: n$ is negligible $\}$, then*

1) $\mu^{-1}(0)=W$; *2)* $\{a_n\}_{n=1}^{\infty}\subseteq W\Longrightarrow\mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty}a_m\Big)=0$.

Proof 1) For any $w\in W$, there exists $a\in A$, such that
$(\mu(a)=0\quad\&\quad w\le a)\Longrightarrow(0\le\mu(w)\le\mu(a)=0)\Longrightarrow(\mu(w)=0)\Longrightarrow(w\in\mu^{-1}(0))$
$\Longrightarrow(W\subseteq\mu^{-1}(0))$;
It is obvious that $W\subseteq\mu^{-1}(0)$. So $\mu^{-1}(0)=W$.

$$2)\quad \mu\Big(\bigvee_{n=1}^{\infty}\bigwedge_{m=n}^{\infty} a_m\Big) \le \mu\Big(\bigvee_{n=1}^{\infty} a_n\Big) \le \sum_{n=1}^{\infty}\mu(a_n) = \sum_{n=1}^{\infty} 0 = 0. \qquad \square$$

Theorem 2.4 $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$.

Proof It is easy to verify that

$$a \bigvee b = (a \bigwedge b')\bigvee(a \bigwedge b)\bigvee(a' \bigwedge b), a = (a \bigwedge b')\bigvee(a \bigwedge b), b = (a \bigwedge b)\bigvee(a' \bigwedge b),$$

and $(a \wedge b') \wedge (a \wedge b) = (a' \wedge b) \wedge (a \wedge b) = (a' \wedge b) \wedge (a \wedge b') = 0$.

Hence, $\mu(a \vee b) = \mu(a \wedge b') + \mu(a \wedge b) + \mu(a' \wedge b) = \mu(a) + \mu(a' \wedge b)$, and consequently,

$$\mu(a \bigvee b) + \mu(a \bigwedge b) = \mu(a) + \mu(a' \bigwedge b) + \mu(a \bigwedge b) = \mu(a) + \mu(b). \quad \square$$

Theorem 2.5 *Define* $d(x,y) = \mu(x \wedge y') + \mu(x' \wedge y), (x, y \in A)$, *then*

1) (A, d) *is a quasi-metric space :* $d(x,x) = 0; d(x,y) = d(y,x); d(x,z) \le d(x,y) + d(y,z)$.

2) If μ *is strictly monotone , then* $(A\ ,\ d)$ *is a metric space :* (A, d) *is a quasi-metric space , and* $d(x, y) = 0$ *if and only if* $x = y$.

Proof 1) Obviously, $d(x,x) = 0, d(x,y) = d(y,x)$. To prove the trigonometric inequality, we first prove the following equation

$$d(x,y) = d(a \bigvee x, a \bigvee y) + d(a \bigwedge x, a \bigwedge y), (\forall a, x, y \in A).$$

In fact ,

$$\begin{aligned}
& d(a \bigvee x, a \bigvee y) + d(a \bigwedge x, a \bigwedge y) \\
&= \mu[(a \bigvee x)\bigwedge(a \bigvee y)'] + \mu[(a \bigwedge x)' \bigwedge (a \bigwedge y)] + \\
&\quad \mu[(a \bigvee x)' \bigwedge (a \bigvee y) + \mu[(a \bigwedge x)\bigwedge(a \bigwedge y)'] \\
&= \mu\{[(a \bigvee x)\bigwedge(a \bigvee y)']\bigvee[(a \bigwedge x)' \bigwedge(a \bigwedge y)]\} + \\
&\quad \mu[(a \bigvee x)' \bigwedge(a \bigvee y)\bigvee[(a \bigwedge x)\bigwedge(a \bigwedge y)']\} \\
&= \mu\{a' \bigwedge[(x \bigwedge y')\bigvee(x' \bigwedge y)]\} + \mu\{a \bigwedge[(x \bigwedge y')\bigvee(x' \bigwedge y)]\} \\
&= \mu\{[a' \bigvee a]\bigwedge[(x \bigwedge y')\bigvee(x' \bigwedge y)]\} \\
&= \mu[(x \bigwedge y')\bigvee(x' \bigwedge y)] = \mu(x \bigwedge y') + \mu(x' \bigwedge y) = d(x,y).
\end{aligned}$$

Thus,

$$\begin{aligned}
d(x,y) + d(y,z) &= d(x \bigvee y, y) + d(x \bigwedge y, y) + d(y \bigvee z, y) + d(y \bigwedge z, y) \\
&\ge d(x \bigvee y \bigvee z, y \bigvee z) + d(y, x \bigwedge y) + d(x \bigvee y \bigvee z, x \bigvee y) + d(y \bigwedge z, y) \\
&\ge \mu[(x \bigwedge z')\bigvee y] + \mu[(z \bigwedge x')\bigvee y] \\
&\ge \mu(x \bigwedge z') + \mu(z \bigwedge x') \\
&= d(x,z)
\end{aligned}$$

2) It suffices to prove that $d(a,b)=0 \Longrightarrow a=b$.

If $a \neq b$, then $a \bigwedge b', a' \bigwedge b$ can not all be 0 . Ortherwise ,

$$b=(a \bigwedge b') \bigvee b = a \bigvee b \Longrightarrow a \leq b, a=(a' \bigwedge b) \bigvee a = b \bigvee a \Longrightarrow b \leq a \Longrightarrow a=b,$$

which is a contradiction.

Hence , $d(a,b)=\mu(a \bigwedge b')+\mu(a' \bigwedge b)>0$, a contradiction to $d(a,b)=0$. □

Remark 2.1 It is easily seen from the arguments of the above that all results of this paper can be generalized to the case of a monoid-valued measure .

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群元值测度的一个定义和一些性质

杨 义 川

(北京工业大学应用数学系，北京 100022)

摘 要: 文中给出了定义在第二可数完备布尔代数上，取值于半序交换群内的群元值测度的一个定义和一些性质.