Journal of Mathematical Research & Exposition
Vol.21, No.1, 81–85, February, 2001

Note on Some Oscillation Theorems in a Recent Paper *

PENG Ming-shu[1,2], *GE Wei-gao*[2], *XU Qian-li*[3]

(1. Dept. of Math., Beijing Normal University, 100875, China;
2. Dept. of Appl. Math., Beijing Institute of Technology, 100081, China ;
3. Dept. of Math., Yiyang Teachers' College, Hunan 413049, China)

Abstract: In this paper, some counterexamples are offered to illustrate that some results stated in a recent paper on the oscillatory behavior of solutions of second order nonlinear difference equation are incorrect.

Key words: Oscillation; monotone solution; difference equation.

Classification: AMS(1991) 39A10/CLC O175.7

Document code: A **Article ID:** 1000-341X(2001)01-0081-05

1. Introduction

In a recent papert [1] the authors provided sufficient conditions for the oscillation of all solutions of the perturbed difference equation

$$\triangle(a_{n-1}(\triangle y_{n-1})^{\sigma}) + F(n, y_n) = G(n, y_n, \triangle y_n), n \geq 1, \tag{1}$$

where $0 < \sigma = p/q$ with p even and q odd integers (even/odd) or p and q odd integers (odd/odd), $\{a_n\}$ is an eventually positive real sequence and there exist real sequences $\{q_n\}$, $\{p_n\}$, and a function $f : R \to R$ such that

$$uf(u) > 0 \text{ for all } u \neq 0, \tag{2}$$

$$f(u) - f(v) = g(u, v)(u - v) \text{ for } u, v \neq 0, \tag{3}$$

where $g(u, v)$ is a nonnegative function; and

$$\frac{F(n, u)}{f(u)} \geq q_n, \frac{G(n, u, v)}{f(u)} \leq p_n, \text{ for } u, v \neq 0. \tag{4}$$

In some oscillation theorems of [1] for the case $\sigma=$ (even/odd), incorrect results about the oscillatory behavior of all sollutions of equations (1) are stated. The aim of this paper is

*Received date: 1998-05-15
Foundation item: Supported by NNSF of China (19871005)
Biography: PENG Ming-shu (1969-), male, Ph.D.

to show this fact with some examples.

2. Main results

For simplicity, we list the conditions used in the main results of [1] as follows:

$$\sum^{\infty}(q_n - p_n) = \infty, \tag{5}$$

$$\sum^{\infty}(q_n - p_n) < \infty, \tag{6}$$

$$\lim_{n\to\infty} \sum_{s=n_0}^{\infty} [\frac{1}{a_s} \sum_{t=n_0}^{s} (q_t - p_t)]^{1/\sigma} = \infty, \tag{7}$$

$$\liminf_{n\to\infty} \sum_{s=n_0}^{n} (q_s - p_s) \geq 0, \text{for all large } n_0, \tag{8}$$

$$\sum^{\infty} [\frac{K}{a_s} - \frac{1}{a_s} \sum_{s=n_0}^{n} (q_s - p_s)] = -\infty, \text{for every constant } K \tag{9}$$

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} (q_s - p_s) = \infty, \text{for all large } n_0, \tag{10}$$

$$\limsup_{n\to\infty} \sum_{s=n_0}^{n} s(q_s - p_s) = \infty, \text{for all large } n_0, \tag{11}$$

$$\sum^{\infty}(q_n - p_n)R(n, n_0) = \infty, \text{where } R(n, n_0) = \sum_{n_0}^{n} \frac{1}{a_s}, \tag{12}$$

$$\sum^{\infty}(q_n - p_n)T(n, n_0) = \infty, \text{where } T(n, n_0) = \sum_{n_0}^{n-1} \frac{1}{a_s}, \tag{13}$$

$$\frac{a_n}{a_{n-1}} \leq 1, \text{for } n \geq 1. \tag{14}$$

The starting point of this paper is the following results proved in [1] as Theorem 2.1(b), Corollary 2.3(b), Theorem 2.4(b), Theorem 2.6(b), Corollary 2.8(b), Corollary 2.10(b), Corollary 2.12 (b):

Theorem 2.1 *Suppose that (5) holds. If* σ *=(even/odd), then every solution* $\{y_n\}$ *of (1) is either oscillatory or* $\{\triangle y_n\}$ *is oscillatory.*

Theorem 2.2 *Suppose that (6) and (7) hold. If* σ *=(even/odd), then every bounded solution* $\{y_n\}$ *of (1) is either oscillatory or* $\{\triangle y_n\}$ *is oscillatory.*

Theorem 2.3 *Suppose that (8) and (9) hold. If* σ *=(even/odd), then the conclusion of Theorem 2.2 follows.*

Theorem 2.4 *Suppose that (10) holds. If* σ *= (even/odd), then the conclusion of*

Theorem 2.1 follows.

Theorem 2.5 *Suppose that $a_n \equiv 1, \sigma \geq 1$ and (11) hold. If σ =(even/odd), then the conclusion of Theorem 2.2 follows.*

Theorem 2.6 *Suppose that $\sigma > 1$ and (12) hold. If σ =(even/odd), then the conclusion of Theorem 2.2 follows.*

Theorem 2.7 *Suppose that $\sigma \geq 1$, (13) and (14) hold. If σ =(even/odd), then the conclusion of Theorem 2.2 follows.*

Now we offer some counterexamples to the above oscillation theorems as follows:

Example 2.1 Consider the difference equation

$$\triangle(n(\triangle y_{n-1})^{\sigma}) + y_n(b(n,y_n) + \theta_n) = b(n,y_n)y_n, n \geq 2,$$

where σ=(even/odd) and $b(n,y_n)$ is any function of n and y_n, which also has been considered in [1].Let $\theta_n = 1/n$,choosing $f(y_n) = y_n$, we have

$$\frac{F(n,y_n)}{f(y_n)} = b(n,y_n) + \frac{1}{n} \geq b(n,y_n) + \frac{1}{2n} \equiv q_n,$$

and

$$\frac{G(n,y_n,\triangle y_n)}{f(y_n)} = b(n,y_n) \leq b(n,y_n) + \frac{1}{4n} \equiv p_n,$$

Therefore we obtain $\sum^{\infty}(q_n - p_n) = \sum^{\infty}\frac{1}{4n} = \infty$, i.e., condition (5) (and also (10)) holds. So according to Theorem 2.1 (Theorem 2.1 (b)[1]), a solution $\{y_n\}$ should be oscillatory or $\{\triangle y_n\}$ should be oscillatory. But in fact, this equation has a solution given by $y = -n$. Neither $\{y_n\}$ nor $\{\triangle y_n\}$ is oscillatory.

Example 2.2 The difference equation

$$\triangle(\frac{1}{n^{1/3}(n-1)^{1/3}}(\triangle y_{n-1})^{2/3}) + y_n(b(n,y_n) + \frac{2}{(n+1)(n-1)}) = b(n,y_n)y_n, n \geq 2,$$

where $b(n,y_n)$ is any function of n and y_n, has a bounded solution by $y_n = \frac{1}{n}$, which is neither oscillatory nor $\{\triangle y_n\}$ is oscillatory, i.e. the conclusion of theorem 2.2 is violated. But we observed that every condition of Theorem 2.2 is satisfied. In fact, by taking $f(y_n) = y_n$, we have

$$\frac{F(n,y_n)}{f(y_n)} = b(n,y_n) + \frac{2}{(n+1)(n-1)} \geq b(n,y_n) + \frac{2}{(n+1)n} \equiv q_n,$$

and

$$\frac{G(n,y_n,\triangle y_n)}{f(y_n)} = b(n,y_n) \leq b(n,y_n) + \frac{1}{(n+1)n} \equiv p_n,$$

Therefore we obtain

$$\sum_{k=n}^{\infty}(q_k - p_k) = \sum_{k=n}^{\infty}\frac{1}{(k+1)k} = \frac{1}{n} < \infty,$$

and $\lim_{n\to\infty}\inf\sum_{s=n_0}^{n}(q_s-p_s)\geq 0$, for all large n_0,

$$\sum_{s=n_0}^{\infty}[\frac{1}{a_s}\sum_{t=s+1}^{\infty}(q_t-p_t)]^{1/\sigma}=\sum_{s=n_0}^{\infty}[\frac{1}{s^{\frac{1}{3}}(s-1)^{\frac{1}{3}}\cdot\frac{1}{s}}]^{3/2}$$
$$=\sum_{s=n_0}^{\infty}\frac{(s-1)^{1/2}}{s}\geq\sum_{s=n_0}^{\infty}\frac{1}{s}=\infty,$$

i.e., condition (6) and (7) hold.

Example 2.3 Consider the nonlinear difference equation

$$\triangle(n^3(n-1)^2(\triangle y_{n-1})^2)+y_n^5(b(n,y_n)+n^5)=b(n,y_n)y_n^5, n\geq 2 \tag{16}$$

where $b(n,y_n)$ is any function of n and y_n. We claim that all conditions of Theorem 2.3 (Theorem 2.4(b)[1]) are satisfied. But the conclusion of theorem 2.3 does not follows. Because this equation has a monotone solution given by $y_n=-\frac{1}{n}$. Now we verify that conditions (8) and (9) hold. By taking $f(y_n)=y_n^5$, we get

$$\frac{F(n,y_n)}{f(y_n)}=b(n,y_n)+n^5\geq b(n,y_n)+\frac{n^5}{2}\equiv q_n,$$

and

$$\frac{G(n,y_n,\triangle y_n)}{f(y_n)}=b(n,y_n)\leq b(n,y_n)+\frac{n^5}{4}\equiv p_n.$$

So we have

$$\sum^{\infty}[\frac{K}{a_s}-\frac{1}{a_s}\sum_{t=n_0}^{s}(q_t-p_t)]=\sum^{\infty}[\frac{K}{s^3(s-1)^2}-\frac{1}{s^3(s-1)^2}\sum_{t=n_0}^{s}\frac{t^5}{4}]$$
$$\leq\sum^{\infty}\frac{K}{s^3(s-1)^2}-\sum^{\infty}\frac{1}{s^3(s-1)^2}\cdot\frac{s^5}{4}$$
$$=\sum^{\infty}\frac{K}{s^3(s-1)^2}-\sum^{\infty}\frac{s^2}{4(s-1)^2}$$
$$=-\infty,$$

i.e., conditions (8) and (9) hold.

Example 2.4 Consider the difference equation

$$\triangle((\triangle y_{n-1})^2)+y_n^3(b(n,y_n)+\frac{4n^2}{(n+1)^2(n-1)^2})=b(n,y_n)y_n^3, n\geq 2 \tag{17}$$

where $b(n,y_n)$ is any function of n and y_n. Obviously $a_n\equiv 1$ and $\frac{a_n}{a_{n-1}}\equiv 1$, i.e. condition (14) holds. Now we verify that conditions (11)-(13) hold. By taking $f(y_n)=y_n^3$, we obtain

$$\frac{F(n,y_n)}{f(y_n)}=b(n,y_n)+\frac{4n^2}{(n+1)^2(n-1)^2}\geq b(n,y_n)+\frac{n^2}{(n+1)^2(n-1)^2}\equiv q_n,$$

and

$$\frac{G(n,y_n,\triangle y_n)}{f(y_n)} = b(n,y_n) \le b(n,y_n) + \frac{n^2}{2(n+1)^2(n-1)^2} \equiv p_n.$$

Hence

$$\sum_{s=n_0}^{\infty} s(q_s - p_s) = \sum_{s=n_0}^{\infty} \frac{s^3}{2(s+1)^2(s-1)^2} \ge \sum_{s=n_0}^{\infty} \frac{1}{2(s+1)} - \sum_{s=n_0}^{\infty} \frac{1}{2(s+1)^2} = \infty,$$

i.e., condition (11) holds. In view of

$$R(n,n_0) = \sum_{s=n_0}^{n} \frac{1}{a_s}, T(n,n_0) \sum_{s=n_0}^{n-1} \frac{1}{a_s}, \text{ and } a_{n-1} \equiv 1,$$

we get

$$\begin{aligned}\sum^{\infty}(q_n - p_n)R(n,n_0) &= \sum^{\infty} \frac{n^2(n-n_0)}{2(n+1)^2(n-1)^2} \ge \sum^{\infty} \frac{(n-n_0)}{2(n+1)^2} \\ &= \sum^{\infty} \frac{1}{2(n+1)} - (n_0+1)\sum^{\infty} \frac{1}{2(n+1)^2} = \infty,\end{aligned}$$

and also one can obtain

$$\sum^{\infty}(q_n - p_n)T(n,n_0) = \infty,$$

i.e., conditions (12) and (13) hold. So all conditions of Theorem 2.5 (Corollary 2.8(b)[1]), Theorem 2.6 (Corollary 2.10(b)[1]) and Theorem 7 (Corollary 2.12(b)[1]) are satisfied. But the conclusions of these theorems do not follow. Because equation (2.17) has a monotone solution given by $y_n = 1/n$.

References:

[1] WONG P J Y and AGARWAL R P. *Oscillation theorems and existence of positive monotone solutions for second order nonlinear difference equations* [J]. Math. Comput. Modelling, 1995, **21**(3): 63–84.

关于几个振动定理的注记

彭名书[1,2], 葛渭高[2], 徐千里[3]

(1. 北京师范大学数学系, 1000875; 2. 北京理工大学应用数学系, 100081;
3. 益阳师范专科学校数学系, 湖南 413049)

摘 要: 本文通过反例指出了最近某文献中几个振动定理存在的错误.