非局部反应扩散方程奇摄动问题*

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摘　要：本文研究了一类具有非局部奇摄动反应扩散初始边值问题.在适当的条件下，利用比较定理讨论了问题解的渐近性态.

关键词：反应扩散；奇摄动；渐近性态.

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作者曾在文[6]-[11]中研究了一类反应扩散方程奇摄动问题.今再考虑如下问题：

$$\frac{\partial u}{\partial t}-\varepsilon^2 Lu=f(x,u,Tu,\varepsilon),\quad 0<t\leqslant T_0,\ x\in\Omega, \tag{1}$$

$$B[u]\equiv\left[a\frac{\partial u}{\partial n}+u\right]=g(x,\varepsilon),\quad (a>0),x\in\partial\Omega, \tag{2}$$

$$u(0,x,\varepsilon)=A(x,\varepsilon), \tag{3}$$

其中

$$L=\sum_{i,j=1}^{n}\alpha_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{n}\beta_i(x)\frac{\partial}{\partial x_i},$$

$$\sum_{i,j=1}^{n}\alpha_{ij}(x)\xi_i\xi_j\geqslant\lambda\sum_{i=1}^{n}\xi_i^2,\forall\ \xi_i\in R,\ \lambda>0,$$

$$Tu=\int_\Omega K(x,y)u(t,y,\varepsilon)\mathrm{d}y,\ x,y\in\Omega,$$

ε 为正的小参数，$x=(x_1,x_2,\cdots,x_n)\in\Omega$，$\Omega$ 为 n 维欧氏空间的有界域，$\partial\Omega$ 为 Ω 的光滑边界，L 为 Ω 上的一致椭圆型算子，$\partial/\partial n$ 为 $\partial\Omega$ 上外法向导数.

问题(1)-(3)是非局部奇摄动问题，有其较大的应用背景[1]-[3]. 本文构造了问题(1)-(3)解的渐近展开式，并讨论了其渐近性态.

假设：

$[H_1]$ L 的系数及 f，g，A，K 关于其变元在相应区域内为充分光滑的函数；

$[H_2]$ $f_u(x,v,Tv,\varepsilon)-f_{Tu}(x,u,Tu,\varepsilon)\geqslant b_0>0$，$u\geqslant v$.

先构造(1)-(3)解的形式渐近展开式.原问题的的退化情形为

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$$u_t = f(x,u,Tu,0),\ 0 < t \leqslant T_0,\ x \in \Omega, \tag{4}$$

$$u(0,x) = A(x,0). \tag{5}$$

由假设知，问题(4)，(5)存在唯一的光滑解 U_0.

设问题(1)-(3)的外部解 U 为：

$$U \sim \sum_{i=0}^{\infty} U_i \varepsilon^i. \tag{6}$$

将(6)代入(1)，(3)，把 f 按 ε 的幂展开，合并各式 ε 的同次幂项，令对应项的系数为零，可得：

$$(U_i)_t = f_u(x,U_0,TU_0,0)U_i + f_{Tu}(x,U_0,TU_0,0)TU_i + F_i + LU_{i-2},$$

$$U_i(0,x) = A_i(x).$$

上面和下面出现带负下标的项设其为零，F_i 为 $U_1,U_2,\cdots,U_{i-1}$ 逐次已知的函数，其结构从略，而 $A_i = \frac{\partial^i A}{\partial \varepsilon^i}]|_{\varepsilon=0}$，$i=1,2,\cdots$.

由上述线性问题的解 U_i，连同退化问题(4)，(5)的解 U_0，代入(6)，便可得到原问题的外部解.但它未必满足边界条件(2)，故还需构造"边界层校正项"V.

先在 $\partial\Omega$ 的邻域建立非奇局部坐标(ρ,φ)[6].定义在 $\partial\Omega$ 的邻域每一点 Q 的坐标为：$\rho(\leqslant\rho_0)$ 为由点 Q 到边界 $\partial\Omega$ 的距离，其中 ρ_0 为足够小，使得 $\partial\Omega$ 上的内法线在所考虑的邻域内互不相交；$\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_{n-1})$ 为 $(n-1)$维流形 $\partial\Omega$ 上的非奇坐标系，点 Q 的坐标 φ 与对应的内法线和 $\partial\Omega$ 交点 P 的坐标 φ 相同.

在 $\partial\Omega$ 的邻域 $0\leqslant\rho\leqslant\rho_0$ 内，有：

$$L = a_{nn}\frac{\partial^2}{\partial\rho^2} + \sum_{i=1}^{n-1} a_{ni}\frac{\partial^2}{\partial\rho\partial\varphi_i} + \sum_{i,j=1}^{n-1} a_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j} + b_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial\varphi_i}, \tag{7}$$

其中

$$a_{nn} = \sum_{i=1}^{n}\left(\frac{\partial\rho}{\partial x_j}\right)^2, a_{ni} = 2\sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}\frac{\partial\varphi_i}{\partial x_j},\ a_{ij} = \sum_{k=1}^{n}\frac{\partial\varphi_i}{\partial x_k}\frac{\partial\varphi_j}{\partial x_k}$$

$$b_n = \sum_{j=1}^{n}\frac{\partial^2\rho}{\partial x_j^2},\ b_i = \frac{\partial^2\varphi_i}{\partial x_j^2}.$$

在 $0\leqslant\rho\leqslant\rho_0$ 上引入多重尺度坐标[4]：

$$\tau = \frac{h(\rho,\varphi)}{\varepsilon}, \bar{\rho} = \rho,\ \varphi = \varphi, \tag{8}$$

其中 $h(\rho,\varphi)$为待定函数，它将由下文的(13)确定.故

$$\frac{\partial}{\partial\rho} = \frac{h_\rho}{\varepsilon}\frac{\partial}{\partial\tau} + \frac{\partial}{\partial\rho},\quad \frac{\partial^2}{\partial\rho^2} = \frac{h_\rho^2}{\varepsilon^2}\frac{\partial^2}{\partial\tau^2} + 2\frac{h_\rho}{\varepsilon}\frac{\partial^2}{\partial\tau\partial\rho} + \frac{\partial^2}{\partial\rho^2}.$$

为方便起见，下面仍用 ρ 来表示 $\bar{\rho}$.由(8)有

$$L = \frac{1}{\varepsilon^2}K_0 + \frac{1}{\varepsilon}K_1 + K_2, \tag{9}$$

其中

$$K_0 = a_{nn}h_\rho^2\frac{\partial^2}{\partial\tau^2},$$

$$K_1 = 2a_{nn}h_\rho \frac{\partial^2}{\partial\tau\partial\rho} + \sum_{i=1}^{n-1} a_{ni}h_\rho \frac{\partial^2}{\partial\tau\partial\varphi_i} + (a_{nn}h_{\rho\rho} + b_n h_\rho)\frac{\partial}{\partial\tau},$$

$$K_2 = a_{nn}\frac{\partial^2}{\partial\rho^2} + \sum_{i=1}^{n-1} a_{ni}\frac{\partial^2}{\partial\rho\partial\varphi_i} + \sum_{i,j=1}^{n-1} a_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j} + b_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial\varphi_i}.$$

设原问题(1)-(3)的解为

$$u = U + V. \tag{10}$$

将它代入(1)得

$$V_t - \varepsilon^2 LV = f(\rho,\varphi,U+V,T(U+V),\varepsilon) - f(\rho,\varphi,U,TU,\varepsilon). \tag{11}$$

并设

$$V \sim \sum_{i=0}^{\infty} v_i(t,\tau,\rho,\varphi)\varepsilon^i. \tag{12}$$

将 (10),(6),(12)代入(11),并将(11)的右端按 ε 的幂展开,比较(11)的 ε 同次幂的系数,可得:

$$(v_0)_t - K_0 v_0 = f(\rho,\varphi,U_0+v_0,T(U_0+v_0),0) - f(\rho,\varphi,U_0,TU_0,0),$$

$$(v_i)_t - K_0 v_i - f_u(\rho,\varphi,U_0+v_0,T(U_0+v_0),0)v_i - f_{Tu}(\rho,\varphi,U_0+v_0,T(U_0+v_0),0)Tv_i$$
$$= \overline{F}_i - K_1 v_{i-1} - K_2 v_{i-2},\quad i=1,2,\cdots,$$

其中 $\overline{F}_i$ 为 $U_0,U_1,\cdots,U_i$ 和 $v_0,v_1,\cdots,v_{i-1}$ 逐次已知的函数,其结构从略.

令

$$h(\rho,\varphi) = \int_0^\rho \frac{d\rho}{\sqrt{a_{nn}}}, \tag{13}$$

故:

$$(v_0)_t - (v_0)_{\tau\tau} = f(\rho,\varphi,U_0+v_0,T(U_0+v_0),0) - f(\rho,\varphi,U_0,TU_0,0), \tag{14}$$

$$(v_i)_t - (v_i)_{\tau\tau} - f_u(\rho,\varphi,U_0+v_0,T(U_0+v_0),0)v_i - f_{Tu}(\rho,\varphi,U_0+v_0,T(U_0+v_0),0)Tv_i$$
$$= \overline{F}_i - K_1 v_{i-1} - K_2 v_{i-2}.\ i=1,2,\cdots. \tag{15}$$

将(10)代入(2),考虑到多重尺度变换(8)得:

$$(V_\tau)_{\tau=\rho=0} = \varepsilon(\sqrt{a_{nn}}/a)(g - B[U]_{\rho=0} - [aV_\rho + V]_{\tau=\rho=0}).$$

由上式并注意到关系式(6),(12)

$$[(V_0)_\tau]_{\tau=\rho=0} = 0, \tag{16}$$

$$[(V_i)_\tau]_{\tau=\rho=0} = (\sqrt{a_{nn}}/a)(g_i - B[U_{i-1}]_{\rho=0} - [a(v_{i-1})_\rho + v_{i-1}]_{\tau=\rho=0}),\quad i=1,2,\cdots, \tag{17}$$

其中 $g_i = [\frac{\partial^i g}{\partial\varepsilon^i}]|_{\varepsilon=0}$, $i=1,2,\cdots$.

由(14),(16)和(15),(17),分别可得具有边界层性质的解 v_i[5].

再设 $\bar{v}_i = \psi(\rho)v_i$, $i=0,1,2,\cdots$,其中 $\psi(\rho)$为在 $0\leqslant\rho\leqslant\rho_0$上充分光滑的函数,并满足

$$\psi(\rho) = \begin{cases} 1,\ 0\leqslant\rho\leqslant(1/3)\rho_0, \\ 0,\ \rho\geqslant(2/3)\rho_0. \end{cases}$$

这时便得到原问题(1)-(3)解的形式渐近展开式

$$u \sim \sum_{i=0}^{\infty}(U_i + \bar{v}_i)\varepsilon^i, 0 < \varepsilon \ll 1. \tag{18}$$

下面来讨论渐近展开式(18)的一致有效性[4].

定理 在假设$[H_1]$-$[H_2]$下,反应扩散方程奇摄动问题(1)－(3),在$[0,T_0]\times(\Omega+\partial\Omega)$上具有形如(18)关于$\varepsilon$一致有效的渐近解.

证明 作辅助函数α,β:

$$\alpha = Y_m - r\varepsilon^{m+1}, \tag{19}$$

$$\beta = Y_m + r\varepsilon^{m+1}, \tag{20}$$

其中r为适当大的待定正常数,而$Y_m \equiv \sum_{i=0}^{m} U_i\varepsilon^i + \sum_{i=0}^{m+2}\bar{v}_i\varepsilon^i$. 显然,

$$\alpha \leqslant \beta,\ t\in[0,T_0],\ x\in\Omega+\partial\Omega. \tag{21}$$

由假设,不难看出,存在正常数M_1,成立:

$$\begin{aligned}B[\alpha] &= B[Y_m] - B[r\varepsilon^{m+1}] \leqslant \sum_{i=0}^{m} B[U_i]\varepsilon^i + \sum_{i=0}^{m+2} B[\bar{v}_i]\varepsilon^i - r\varepsilon^{m+1}\\ &\leqslant \sum_{i=1}^{m+2}\{(\alpha/\sqrt{a_{nn}})[(v_i)_\tau]_{\tau=\rho=0} - (g_{i-1} - B[U_{i-1}]_{\rho=0} -\\ &\quad [(v_{i-1})_\rho + v_{i-1}]_{\tau=\rho=0})\}\varepsilon^i + (M_1 - r)\varepsilon^{m+1}\\ &\leqslant (M_1 - r)\varepsilon^{m+1}.\end{aligned}$$

仅需选取$r\geqslant M_1$,就有$B[\alpha]\leqslant 0,\ x\in\partial\Omega$. 同理可证$B[\beta]\geqslant 0,\ x\in\partial\Omega$. 即有

$$B[\alpha] \leqslant 0 \leqslant B[\beta],\ x\in\partial\Omega. \tag{22}$$

下面来证明:

$$\alpha(0,x,\varepsilon) \leqslant A(x,\varepsilon) \leqslant \beta(0,x,\varepsilon),\ x\in\Omega+\partial\Omega, \tag{23}$$

$$(\alpha)_t - \varepsilon^2 L\alpha - f(x,\alpha,T\alpha,\varepsilon) \leqslant 0,\ x\in\Omega, \tag{24}$$

$$(\beta)_t - \varepsilon^2 L\beta - f(x,\beta,T\beta,\varepsilon) \geqslant 0,\ x\in\Omega. \tag{25}$$

(i) 当$\rho\geqslant(2/3)\rho_0$时,因$\bar{v}_i=0$,这时$Y_m \equiv \sum_{i=0}^{m} U_i\varepsilon^i$,再由假设,存在正常数$M_2$,成立:

$$\alpha|_{t=0} = Y_m|_{t=0} - r\varepsilon^{m+1} = \sum_{i=0}^{m}[U_i|_{t=0}]\varepsilon^i - r\varepsilon^{m+1} \leqslant A(x,\varepsilon) + (M_2 - r)\varepsilon^{m+1}.$$

故再选取$r\geqslant M_2$,就有$\alpha|_{t=0}\leqslant A(x,\varepsilon),\ x\in\Omega$. 同理可证:$\beta|_{t=0}\geqslant A(x,\varepsilon),\ x\in\Omega$. 即(23)成立.

由假设,对足够小的$0<\varepsilon\leqslant\varepsilon_1(\leqslant\varepsilon_0)$,存在正常数$M_3$,有:

$$\begin{aligned}&(\alpha)_t - \varepsilon^2 L\alpha - f(x,\alpha,T\alpha,\varepsilon)\\ &= (Y_m - r\varepsilon^{m+1})_t - \varepsilon^2 L[Y_m - r\varepsilon^{m+1}] - f(x,Y_m - r\varepsilon^{m+1},T(Y_m - r\varepsilon^{m+1}),\varepsilon)\\ &= (Y_m)_t - \varepsilon^2 LY_m - f(x,Y_m,TY_m,\varepsilon) +\\ &\quad [f(x,Y_m,TY_m,\varepsilon) - f(x,Y_m - r\varepsilon^{m+1},T(Y_m - r\varepsilon^{m+1}),\varepsilon)]\\ &\leqslant [(U_0)_t - f(x,U_0,TU_0,0)] +\\ &\quad \sum_{i=1}^{m}[(U_i)_t - f_u(x,U_0,TU_0,0)U_i - f_{Tu}(x,U_0,TU_0,0)TU_i - F_i - LU_{i-2}]\varepsilon^i +\\ &\quad M_3\varepsilon^{m+1} - b_0 r\varepsilon^{m+1} \leqslant (M_3 - b_0 r)\varepsilon^{m+1}.\end{aligned}$$

选取$r\geqslant M_3/b_0$,不等式(24)成立.

同理可证不等式(25)也成立.

(ii) 当 $(1/3)\rho_0<\rho<(2/3)\rho_0$ 时，由于 v_i 为具有边界层性质的函数，它在 $(1/3)\rho_0<\rho<(2/3)\rho_0$ 上关于 ε^{m+1} 为高阶小量，故可仿照 (i) 的方法证明关系式(23)—(25)成立.

(iii) 当 $\rho\leqslant(1/3)\rho_0$ 时，因 $\bar{v}_i=v_i$，这时

$$Y_m\equiv\sum_{i=0}^{m}U_i\varepsilon^i+\sum_{i=0}^{m+2}v_i\varepsilon^i,$$

再由假设，存在正常数 M_2，成立：

$$\alpha|_{t=0}=Y_m|_{t=0}-r\varepsilon^{m+1}=\sum_{i=0}^{m}[U_i|_{t=0}]\varepsilon^i+\sum_{i=0}^{m+2}[v_i|_{t=0}]\varepsilon^i-r\varepsilon^{m+1}$$
$$\leqslant A(x,\varepsilon)+(M_2'-r)\varepsilon^{m+1}.$$

故选取 $r\geqslant M'_2$，就有 $\alpha|_{t=0}\leqslant A(x,\varepsilon)$，$x\in\Omega$. 同理可证：$\beta|_{t=0}\geqslant A(x,\varepsilon)$，$x\in\Omega$. 即(23)成立.

由假设，对足够小的 $0<\varepsilon\leqslant\varepsilon'_1(\leqslant\varepsilon_1)$，存在正常数 M_3'，有：

$$\begin{aligned}
&(\alpha)_t-\varepsilon^2L\alpha-f(x,\alpha,T\alpha,\varepsilon)\\
&=(Y_m-r\varepsilon^{m+1})_t-\varepsilon^2L[Y_m-r\varepsilon^{m+1}]-f(x,Y_m-r\varepsilon^{m+1},T(Y_m-r\varepsilon^{m+1}),\varepsilon)\\
&=(Y_m)_t-\varepsilon^2LY_m-f(x,Y_m,TY_m,\varepsilon)+\\
&\quad[f(x,Y_m,TY_m,\varepsilon)-f(x,Y_m-r\varepsilon^{m+1},T(Y_m-r\varepsilon^{m+1}),\varepsilon)]\\
&\leqslant[(U_0)_t-f(x,U_0,TU_0,0)]+\\
&\quad\sum_{i=1}^{m}[(U_i)_t-f_u(x,U_0,TU_0,0)U_i-f_{Tu}(x,U_0,TU_0,0)TU_i-F_i-LU_{i-2}]\varepsilon^i+\\
&\quad[(v_0)_t-(v_0)_{\tau\tau}-f(\rho,\varphi,U_0+v_0,T(U_0+v_0),0)+f(\rho,\varphi,U_0,TU_0,0)]+\\
&\quad\sum_{i=1}^{m+2}[(v_i)_t-(v_i)_{\tau\tau}-f_u(\rho,\varphi,U_0+v_0,0)v_i-\\
&\quad f_{Tu}(\rho,\varphi,U_0+v_0,0)Tv_i-F_i+K_1v_{i-1}+K_2v_{i-2}]\varepsilon^i+M_3'\varepsilon^{m+1}-b_0r\varepsilon^{m+1}\\
&\leqslant(M_3'-b_0r)\varepsilon^{m+1}.
\end{aligned}$$

选取 $r\geqslant M_3'/b_0$，不等式(24)成立.

同理可证不等式(25)也成立.

故仅需选择足够大的 r，关系式(21)—(25)成立. 由此得到[5]：

$$\alpha(t,x,\varepsilon)\leqslant u(t,x,\varepsilon)\leqslant\beta(t,x,\varepsilon),\quad(t,x,\varepsilon)\in[0,T]\times(\Omega+\partial\Omega)\times[0,\varepsilon_1'].$$

由(19)，(20)知，

$$u=\sum_{i=0}^{m}U_i\varepsilon_i+\sum_{i=0}^{m+2}\bar{v}_i\varepsilon^i+O(\varepsilon^{m+1}),0<\varepsilon\ll1.$$

即关系式(18)成立. 定理证毕.

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Singularly Perturbed Problem for Nonlocal Reaction Diffusion Equations

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Abstract: The initialboundary value problems for the nonlocal singularly perturbed reaction diffusion equations are considered. Under suitable conditions, using the comparison theorem the asymptotic behavior of solution for the intiial boundary value problems are studied.
Key words: reaction diffusion; singular perturbation; asymptotic behavior.