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CoHopficity of Self-Injective Rings *

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Abstract: It is shown that a self-injective ring R is coHofian ring if and only if R has stable range one. This answers the open problem 5 of Varadarajian in [9] for self-injective ring R,i.e.,$M_n(R)$ is coHopfian for coHopfian ring R. As a consequence of we answer problem of Goodeal in the affirmative in [3], for self-injective regular rings.

Key words: coHopfian rings; self-injective rings; stable range one.

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1. Introduction

Let R be an associative ring with identity.The concept of coHopfian rings was introduced by Roiterg in algebraic topology[6] and named by Varadarajan[9]. A ring R is called coHopfian in Mod-R if for every injective homomorphism $f: R \to R$ is an isomorphism. One question on coHopfian rings is proposed by K. Varadarajan[9].

Question 1 Characterize the coHopfian ring R for which the matrix ring over R is also a coHopfian in mod-$M_n(R)$.

In this paper, we will answer the Varadarajan's question in the case when R is a self-injective ring. In particular, we prove that "directly finite regular " properly is preserved under the formation of matrix, provided that the ring is a self-injective regular ring. It give a answer to an open problem raised by Goodeal[3] that whether "directly finite regular" prserved under the formation of matrix rings.

Thoughout this paper, we use ann(x) to denote the right annihilator of x in R.For more information on coHopfian rings and list of examples, see [9].

We begin with a basic lemma.

Lemma 2 *Let R be a self-injective ring.Then* $\mathrm{ann}(a) \subset \mathrm{ann}(b)$ *if and only if* $b = ax$.

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Proof ($\Longrightarrow$):Trivial.

($\Longleftarrow$): Define a map $f : aR \to R$ by $f(ar) = br$,for $r \in R$. Since $\mathrm{ann}(a) \subset \mathrm{ann}(b)$, f is well-defined. Now we have the following commutative diagram:

$$\begin{array}{ccccc} & & R & & \\ & & f\uparrow & \nwarrow g & \\ 0 & \to & aR & \xrightarrow{i} & R, \end{array}$$

where i is an inclusion morphism. Since R is injective module, there exists $g : R \to R$ such that $g \cdot i = f$.Denote $x = g(1)$. Then $b = f(a \cdot 1) = g \cdot i(a \cdot 1) = ag(1) = ax$ as required. □
The following characterization of a coHopfian ring was given by Varadarajan[9].

Lemma 3 *Let R be a ring. R is coHopfian if and only if a is an invertible element for* $\mathrm{ann}(a) = 0$.

Recall that R is said to have stable range one if for any $a, b \in R$ satisfying $aR+bR = R$, there exists $y \in R$ such that $a + by$ is a unit.

The main result of this note is

Theorem 4 *Let R be a self-injective ring.R is coHopfian in mod-R ring if and only if R has stable range one.*

Proof ($\Longrightarrow$): Assume that $aR + bR = R$. Then $\mathrm{ann}(a) \cap \mathrm{ann}(b) = 0$. Denote f : $\mathrm{ann}(b) \to R$ by $f(x) = xa$ for $x \in \mathrm{ann}(b)$. Since $\mathrm{ann}(a) \cap \mathrm{ann}(b) = 0$, f is well-defined and a monomorphism. From the injectity of R, there exists $\theta : R \to R$ such that the following diagram commutates:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{ann}(b) & \xrightarrow{f} & R & \xrightarrow{\pi} & R/\mathrm{Im}f & \to & 0 \\ & & \uparrow 1 & & \uparrow \theta & & \uparrow \psi & & \\ 0 & \to & \mathrm{ann}(b) & \xrightarrow{i} & R & \xrightarrow{\pi_b} & bR & \to & 0, \end{array}$$

where i is an inclusion map and $\pi_b(r) = br$. We claim that ψ is a monomorphism. Suppose that $\psi(br_1) = \psi(br_2)$,i.e., $\theta(r_1) + \mathrm{Im}f = \theta(r_1) + \mathrm{Im}f$. Hence $\theta(r_1 - r_2) = f(x_0) \in \mathrm{Im}f, x_0 \in \mathrm{ann}(b)$. It follows that $r_1 - r_2 \in \mathrm{ann}(b)$ and hence $br_1 - br_2 = 0$. Since R is a coHopfian in mod-R, then θ is ismorphism. Thus there exists a unit $y = \theta(1) \in R$ such that $f(r) = \theta i(r)$,i.e., $r \cdot a = r \cdot y$, for any $r \in \mathrm{ann}(b)$. So that $r \cdot a \cdot y^{-1} = r$. Therefore we prove that $\mathrm{ann}(b) \subset \mathrm{ann}(ay^{-1} - 1)$. By Lemma 2, $ay^{-1} - 1 = bx$,i.e.,$ay^{-1} - bx = 1$. Since y is a unit,we get $a - bxy$ is a unit of R.

($\Longleftarrow$): Let $a \in R$ with $\mathrm{ann}(a) = 0$. By Lemma 3,it is sufficient to show that a is invertible. We first show that there exists $b \in R$ such that $ab = 1$. For this purpose, Let $f : Ra \to R$ be defined by $f(ra) = r$. Since $\mathrm{ann}(a) = 0$,f is well-defined. Now we have the following commutative diagram:

$$\begin{array}{ccc} Ra & \xrightarrow{i} & R \\ f\downarrow & \swarrow g & \\ R & & , \end{array}$$

where i is an inclusion map. By the assumption, R is an injective module, there exits $g : R \to R$ such that $g \cdot i = f$. Thus $g \cdot i(a) = f(a)$. This implies $a \cdot g(1) = 1$. Denote $g(1) = b$. Then $ab = 1$. Since R has stable range one, there exists $c \in R$ such that ca is unit. Thus a is invertible as required.This completes the proof of theorem. □

The following lemma guarantee that stable one preserves under the formation of matrix rings.

Lemma 5[10]. *Let R be a ring. R has stable range one if and only if $M_n(R)$ has stable range one.*

Proposition 6 *Let R be a self-injective ring. R is coHopfian in mod-R if and only if $M_n(R)$ is coHopfian in mod-$M_n(R)$.*

Proof It is well-known that $M_n(R)$ is also a self-injective ring. By Theorem 4 and Lemma 5, we have R is coHopfian$\Leftrightarrow$R has stable range one$\Leftrightarrow$ $M_n(R)$ has stable range one$\Leftrightarrow$ $M_n(R)$ is coHopfian. This completes the proof. □

Proposition 6 answers the question of Varadarajan for self-injective rings. In particular, since quasi-Frobeius rings are self-injective rings. We have

Corollary 7 *Let R be a quasi-Frobenius rings. R is coHopfian in mod-R if and only if $M_n(R)$ is coHopfian in mod-$M_n(R)$.*

As a consequence of Proposition 6 and Corollary 7,we have

Remark 8 Let R be one of the following rings:

(1) R is a Frobenius algebra[4];

(2) R is finite-dimensional semisimple algebra(such as $R = M_n(F)$, where F is a field)[4];

(3) $R = SG$, where S is a quasi-Frobensius ring(such as a field) and G is a finite group([1]);

(4) $R = \mathbf{Z}/n\mathbf{Z}$[7];

(5) $R = F[x]/I$, where $F[x]$ is a polynomial ring over field F and I is a nonzero idea[7];

(6) R is semilocal ring which R is a cogenerator of mod-R[1];

(7) $R = \mathrm{End}_R(M)$, where M is quasi-injective right R-module[2];

(8) R is IF rings(i.e., every injective R-modules is flat)[11].

Then R is coHopfian in mod-R if and only if $M_n(R)$ is coHopfian in mod-$M_n(R)$.

Recall that a module M is said to be directly finite if $M \oplus N \cong M$ implies $N = 0$. It is an open problem of Goodeal[3] that whether "directly finite regular " is Morita-invariant. Shepherdson[8] showed that "directly finite" is not Morita-invariant.

Note that direct finiteness involved in the concepts of "finite projections" in operator algebras and "finite idempotents" in Baer rings[5]. As an application of Theorem 4, we show that the problem of Goodeal is in the affirmative for matrix ring over directly finite regular self-injective rings.

Lemma 9 *Let R be a self-injective regular ring. R is coHopfian in mod-R if and only if R is directly finite.*

Proof From Lemma 2 and [3] Lemma 5.1,we get the conclusion.

The following Prosition follows from Lemma 9 and Theorem 4.

Proposition 10 *Let R be a self-injective regular ring. If R is directly finite then $M_n(R)$ is also directly finite for all n.*

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自内射环的余 Hopf 性

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摘 要: 本文证明了自内射环 R 是余 Hopf 的当且仅当 R 满足 stable range one. 于是得到了 Varadarajan 在 [9] 中的公开问题对于自内射环是成立的，即 $M_n(R)$ 是余 Hopf 的当且仅当 R 是余 Hopf 的. 作为应用证明了 Goodeal 的一个公开问题对于自内射正则环有肯定的回答.