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# Regular Congruence Classes of Ordered Semigroups *

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**Abstract:** The aim of this paper is to determine the subset of an ordered semigroup $S$ that can serve as a class of some regular congruences on $S$. The results on semigroups without order given in [5] can be obtained as consequences.

**Key words:** ordered semigroup; regular congruence; filter.

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The problem of which subsets of semigroups can serve as a semilattice congruence classes have been considered by M. Petrich in [5, II.3.14]. Petrich has shown that a subset $C$ of a semigroup $S$ is a $N$-class if and only if $C$ is the intersection of a completely semiprime ideal and a filter. For ordered semigroups, Kehayopulu and Tsingelis [6] have determined the subsets of an ordered semigroup $S$ that can serve as some semilattice congruence classes.

Congruences on ordered semigroups play an important role in studying the structures of ordered semigroups. For any congruence $\rho$ on an ordered semigroup $S$, in general, we do not know whether the quotient semigroup $S/\rho$ is also an ordered semigroup. Even if $S/\rho$ is an ordered semigroup, then are orders on $S/\rho$ relative to the order on the original ordered semigroup $S$. So we have introduced and studied regular congruences on $S$ (see [3], [4]). Based on regular congruences, it is natural to ask that which subset of an ordered semigroup $S$ can serve as classes of some regular congruences on $S$. In this paper, we prove that the results of Petrich mentioned above are true in ordered semigroups, as well. Besides, a semigroup $S$ endowed with the equlity relation $\leq := \{(x, y) \mid x = y\}$ is an ordered semigroup, and in this case congruences on $S$ conicide with regular congruences, so one can easily obtain the results in [5] as applications of the results of this paper.

In the sequal, $S$ always denotes an ordered semigroup. A non-empty subset $I$ of $S$ is called an ideal of $S$ if 1) $IS \subseteq I$, $SI \subseteq I$. 2) $a \in I$, $S \ni b \leq a$ implies $b \in I$ [1, 2]. An ideal

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$I$ of $S$ is called prime if for any $a, b \in S$ such that $ab \in I$, we have $a \in I$ or $b \in I$. $I$ is called semiprime if for any $a \in S$ such that $a^2 \in I$, we have $a \in I$, or equivalent definition: $A \subseteq S$ and $A^2 \subseteq I$ implies that $A \subseteq I$ [2].

A subsemigroup $F$ of $S$ is called a filter of $S$ [4] if 1) $a, b \in S$, $ab \in F$ implies $a \in F$ and $b \in F$. 2) $a \in F, S \ni c \geq a$ implies $c \in F$.

**Definition 1**[4] *Let $S$ be an ordered semigroup, $\rho$ a congruence on $S$. $\rho$ is called regular if there exists an order "$\preceq$" on $S/\rho$ such that:*

*(1) $(S/\rho, \cdot, \preceq)$ is an ordered semigroup (where "$\cdot$" is the usual multiplication on $S/\rho$ defined by $(x)_\rho \cdot (y)_\rho := (xy)_\rho$).*

*(2) The mapping $\varphi : S \to S/\rho | x \to (x)_\rho$ is isotone ( Then $\varphi$ is a homomorphism).*

We denote by "$\mathcal{N}$" the equivalence relation on $S$ defined by

$$\mathcal{N} := \{(x, y) | N(x) = N(y)\},$$

where $N(a)$ is the filter of $S$ generated by $a (a \in S)$. Then $\mathcal{N}$ is the semilattice congruence on $S$ [1]. Note that in [3], we proved $\mathcal{N}$ is the regular semilattice congruence on $S$. we denote by $(x)_\mathcal{N}$ the $\mathcal{N}$-class containing $x$.

For convinience, for $H \subseteq S$, we denote

$$[H) := \{x \in S | x \geq h \text{ for some } h \in H\}, \quad (H] := \{x \in S | x \leq h \text{ for some } h \in H\}.$$

**Theorem 1** *Let $S$ be an ordered semigroup. If every ideal $C$ on the semigroup structure of $S$ is a congruence class of at least one regular congruence on $S$, then $C$ is convex. Conversely, if $(C] = C$, then the reverse statement is true.*

**Proof** Let $C$ be a congruence class of a regular congruence $\rho$, and let $a, b \in C$. For any $c \in S$, if $a \leq c \leq b$, by Definition 1, we get $a\rho \leq c\rho \leq b\rho$. Since $a\rho = b\rho$, so we have $a\rho = c\rho = b\rho$, that is, $c \in C$.

Conversely, Let $\rho_C$ be the Rees congruence about $C$ on $S$. Then it is clear that $C$ is the congruence class of $\rho_C$. So to show that $C$ is the congruence class of a regular congruence on $S$, we need only to show that $\rho_C$ is a regular congruence on $S$. Now define a relation $\preceq$ on the quiotent semigroup $S/\rho_C$ $(= \{\{x\} \mid x \in S \backslash C\} \cup \{C\})$ as follows:

$$\preceq := \{(C, \{x\}) | x \in S \backslash C\} \cup \{(\{x\}, \{y\}) | x, y \in S \backslash C, x \leq y\} \cup \{(C, C)\}.$$

Then $(S/\rho_C, \cdot, \preceq)$ is an ordered semigroup, and $\rho_C$ is regular. In fact:

(1) Let $(x)_{\rho_C} \in S/\rho_C (\Rightarrow (x)_{\rho_C} \preceq (x)_{\rho_C}?)$. Then, $(x)_{\rho_C} = \{x\}$, $x \in S \backslash C$ or $(x)_{\rho_C} = C$.
Let $(x)_{\rho_C} = \{x\}, x \in S \backslash C$. Since $x \leq x$, then $(\{x\}, \{x\}) \in \preceq \Rightarrow (x)_{\rho_C} \preceq (x)_{\rho_C}$.
Let $(x)_{\rho_C} = C$. Since $(C, C) \in \preceq$, we have $(x)_{\rho_C} \preceq (x)_{\rho_C}$.

(2) Let $(x)_{\rho_C} \preceq (y)_{\rho_C}, (y)_{\rho_C} \preceq (x)_{\rho_C} (\Rightarrow (x)_{\rho_C} = (y)_{\rho_C}?)$. Since $(x)_{\rho_C}, (y)_{\rho_C} \in S/\rho_C$, we have $(x)_{\rho_C} = \{x\}, x \in S \backslash C$ or $(x)_{\rho_C} = C$. $(y)_{\rho_C} = \{y\}, y \in S \backslash C$, or $(y)_{\rho_C} = C$. We consider the following two cases:

(a) $(x)_{\rho_C} = \{x\}, x \in S \backslash C$.

$$\begin{aligned}(x)_{\rho_C} \preceq (y)_{\rho_C} \Rightarrow & ((x)_{\rho_C}, (y)_{\rho_C}) \in \{(C, \{z\}) | z \in S \backslash C\} \text{ or} \\ & ((x)_{\rho_C}, (y)_{\rho_C}) \in \{(C, C)\} \text{ or} \\ & ((x)_{\rho_C}, (y)_{\rho_C}) \in \{(\{t\}, \{m\}) | t, m \in S \backslash C, t \leq m\}.\end{aligned}$$

From the first two cases, we have $(x)_{\rho_C} = C$, then $x \in C$, this is impossible. Thus $(x)_{\rho_C} = \{t\}, (y)_{\rho_C} = \{m\}$ for some $t, m \in S\backslash C, t \leq m$. Therefore, $t = x, y = m, x \leq y$.

Since $(y)_{\rho_C} = \{y\}$, and $(y)_{\rho_C} \preceq (x)_{\rho_C}$, we have $y \leq x$. Then $x = y, \{x\} = \{y\}$, and $(x)_{\rho_C} = (y)_{\rho_C}$.

(b) $(x)_{\rho_C} = C$. We consider two subcases:

(i) If $(y)_{\rho_C} = \{y\}, y \in S\backslash C$. Since $(y)_{\rho_C} \preceq (x)_{\rho_C}$, by (a), we have $(x)_{\rho_C} = \{x\}$, a contradiction.

(ii) If $(y)_{\rho_C} = C$. Then $(y)_{\rho_C} = (x)_{\rho_C}$.

(3) Let $(x)_{\rho_C} \preceq (y)_{\rho_C}, (y)_{\rho_C} \preceq (z)_{\rho_C}$ ($\Rightarrow (x)_{\rho_C} \preceq (z)_{\rho_C}$?). Since $(x)_{\rho_C}, (z)_{\rho_C} \in S/\rho_C$, we have $(x)_{\rho_C} = \{x\}, x \in S\backslash C$ or $(x)_{\rho_C} = C$; $(z)_{\rho_C} = \{z\}, z \in S\backslash C$ or $(z)_{\rho_C} = C$. We consider the cases:

(a) Let $(x)_{\rho_C} = \{x\}, x \in S\backslash C$. Since $(x)_{\rho_C} \preceq (y)_{\rho_C}$, by (a) of (2), we have $(y)_{\rho_C} = \{y\}, y \in S\backslash C$, and $x \leq y$. Furthermore, Since $(y)_{\rho_C} = \{y\} \preceq (z)_{\rho_C}$, by (a) of (2), we have $(z)_{\rho_C} = \{z\}, z \in S\backslash C$, and $y \leq z$. Then $x \leq z$, and $x, z \in S\backslash C$. Thus $\{x\} \preceq \{z\}, (x)_{\rho_C} \preceq (z)_{\rho_C}$.

(b) Let $(x)_{\rho_C} = C$. We consider two subcases:

(i) If $(z)_{\rho_C} = \{z\}, z \in S\backslash C$, since $(C, \{z\}) \in \preceq$, then $(x)_{\rho_C} \preceq (z)_{\rho_C}$.

(ii) If $(z)_{\rho_C} = C$, then $(x)_{\rho_C} = (z)_{\rho_C}$.

(4) Let $(x)_{\rho_C}, (y)_{\rho_C}, (z)_{\rho_C} \in S/\rho_C, (x)_{\rho_C} \preceq (y)_{\rho_C}$($\Rightarrow (x)_{\rho_C}(z)_{\rho_C} \preceq (y)_{\rho_C}(z)_{\rho_C}$?). Since $(x)_{\rho_C} \in S/\rho_C$, we have $(x)_{\rho_C} = \{x\}, x \in S\backslash C$ or $(x)_{\rho_C} = C$.

(a) Let $(x)_{\rho_C} = C$. Let $t \in C (C \neq \emptyset)$. Since $(t)_{\rho_C} \in S/\rho_C$, we have $(t)_{\rho_C} = \{t\}, t \in S\backslash C$ or $(t)_{\rho_C} = C$. If $(t)_{\rho_C} = \{t\}, t \in S\backslash C$, then $t \in S\backslash C$, a contradiction. Thus $(t)_{\rho_C} = C$, and $(x)_{\rho_C} = (t)_{\rho_C}$. It implies that $(x)_{\rho_C}(z)_{\rho_C} = (t)_{\rho_C}(z)_{\rho_C} = (tz)_{\rho_C}$. Since $tz \in CS \subseteq C$, we have $(tz)_{\rho_C} = C$. On the other hand, $(C, (x)_{\rho_C}) \in \preceq, \forall (x)_{\rho_C} \in S/\rho_C$ (*).

Let $(x)_{\rho_C} \in S/\rho_C$. Then $(x)_{\rho_C} = \{x\}, x \in S\backslash C$, or $(x)_{\rho_C} = C$. Since $x \in S\backslash C$, we have $(C, \{x\}) \in \preceq$, then $C \preceq (x)_{\rho_C}$. Let $(x)_{\rho_C} = C$, since $(C, C) \in \preceq$, we have $C \preceq (x)_{\rho_C}$. Since $(y)_{\rho_C}(z)_{\rho_C} = (yz)_{\rho_C} \in S/\rho_C$, by (*), we $C \preceq (yz)_{\rho}$. Hence $(x)_{\rho_C}(z)_{\rho_C} \preceq (y)_{\rho_C}(z)_{\rho_C}$.

(b) Let $(x)_{\rho_C} = \{x\}, x \in S\backslash C$($\Rightarrow (xz)_{\rho_C} \preceq (yz)_{\rho_C}$?).
Since $(x)_{\rho_C} \preceq (y)_{\rho_C}$, by (a) of (2), we have $(y)_{\rho_C} = \{y\}, y \in S\backslash C$, and $x \leq y$. Then $xz \leq yz$.

(i) If $xz \in C$, then $(xz)_{\rho_C} \preceq (yz)_{\rho_C}$ (by (*)).

(ii) If $xz \in S\backslash C$, then $yz \in S\backslash C$. Thus $\{xz\} \preceq \{yz\}$. $(xz)_{\rho_C} = \{xz\}$, $(yz)_{\rho_C} = \{yz\}$, and $(xz)_{\rho_C} \preceq (yz)_{\rho_C}$.

In the same way, we can show that the relation "$\preceq$" is left compatible with respect to the multiplication of $S$.

(5) $\rho_C$ is regular. In fact, let $\varphi : S \longrightarrow S/\rho_C | x \rightarrow (x)_{\rho_C}$. If $x \leq y$, we consider three subcases:

(a) $x \in C$. Then $(x)_{\rho_C} = C \preceq (y)_{\rho_C}$.

(b) $y \in C$. Since $C = (C]$, then $C$ is an ideal of $S$. Thus $x \in C$, and $(x)_{\rho_C} = (y)_{\rho_C} = C$.

(c) $x, y \in S\backslash C$. Then we have $(x)_{\rho_C} = \{x\} \preceq \{y\} = (y)_{\rho_C}$. □

**Corollary 1** *If $C$ is an ideal of an ordered semigroup $S$, then $C$ is a congruence class of at least one regular congruence on $S$.*

**Theorem 2** *Let $C$ be a subset of an ordered semigroup $S$. Then $C$ is a congruence class* *of a regular semilattice congruence if and only if $C$ is the intersection of a semiprime ideal and a filter of $S$.*

**Proof** Let $\rho$ be a regular semilattice congruence on $S$, and $\varphi$ a natural homomorphism from $S$ onto $Y = S/\rho$. Then $S = \bigcup S_\alpha$, $\forall \alpha \in Y$, where $S_\alpha$ is an ordered subsemigroup of $S$. By the hypothesis, $C = S_\gamma$ for some $\gamma \in Y$. Let

$$A = \bigcup_{\alpha \leq \gamma} S_\alpha, \quad B = \bigcup_{\alpha \geq \gamma} S_\alpha.$$

Then $C = A \cap B$. We now need only to verify that $A$ is a semiprime ideal of $S$ and $B$ is a filter of $S$.

If $b \leq a \in A, b \in S$, then there exists $\beta \in Y(\beta \leq \gamma)$ such that $a \in S_\beta$. Since $\varphi$ is isotone, then $f(b) \leq f(a) = \beta$. Thus $b \in S_{f(b)} \subseteq A$.

If $a \in A$, then there exists $\beta \in Y$ such that $a \in S_\beta$. Since

$$(\forall s \in S) \;\; f(sa) = f(s)f(a) \leq f(a) = \beta \leq \gamma.$$

Thus $sa \in S_{f(sa)} \subseteq A$. By symmetry, we can show that $as \in A$. Furthermore, it is easily seen that if $a \in S, a^2 \in A$, then $f(a) = f(a^2) \leq \gamma$ implies that $a \in A$. Hence $A$ is a semiprime ideal of $S$.

If $a \geq b \in B, a \in S$, then there exists $\beta \in Y(\beta \geq \gamma)$ such that $b \in S_\beta$. Then $f(a) \geq f(b) = \beta \geq \gamma$. Thus $a \in S_{f(a)} \subseteq B$.

If $ab \in B$, then there exists $\alpha \in Y(\geq \gamma)$ such that $ab \in S_\alpha$. Thus

$$f(a), f(b) \geq f(a)f(b) = f(ab) = \alpha \geq \gamma.$$

Consequently, $a \in S_{f(a)} \subseteq B, b \in S_{f(b)} \subseteq B$. Conversely, if $a, b \in B$, then there exist $\alpha, \beta \in Y$, such that $\gamma \leq \alpha, \beta$, and $a \in S_\alpha, b \in S_\beta$. Clearly, $\gamma \leq \alpha\beta$. Hence $ab \in S_{\alpha\beta} \subseteq B$.

Conversely, let $C = A \cap B$, and $A$ be a semiprime ideal of $S$ and $B$ a filter of $S$. We define a relation on $S$ as follows:

$$\sigma := \{(a, b) \in S \times S \;\mid\; x \in S^1, xa \in C \;\text{ iff }\; xb \in C\}.$$

In the analogous way in the proof of Theorem II.3.12 (see [5]), we have that $\sigma$ is a semilattice congruence on $S$. In order to show that $\sigma$ is regular, it is sufficient to show that if $a \leq b$, $a, b \in S$, then $a\sigma \leq b\sigma$, that is, $(ab)\sigma = a\sigma$. In fact, if $x \in S^1, xab \in C$, that is, $xab \in A$, and $xab \in B$. Since $xa^2 \leq xab$ and $B$ is a filter of $S$, we have $xa^2 \in A, B$. Therefore $xa^2 \in C$. Conversely, if $xa^2 \in C$, then $xa^2 \in A$, and $xa^2 \in B$. Since $xa^2 \leq xab$, we have $xab \in B$. On the other hand, since $xa^2 \in A$, and $A$ is an ideal of $S$, then $(axa)^2 = a(xa^2)xa \in A$. Thus $(ax)^2 \in A$. Since $A$ is semiprime, we have $ax \in A$, and so $(xa)^2 \in A$. Therefore, $xa \in A$, and $xab \in A$. Consequently, $xab \in C$. By the defintion of $\sigma$, we have $(ab)\sigma = a\sigma$. □

**Corollary 2** *Let $S$ be a semilattice. Then a subset $C$ of $S$ is a congruence class of a regular semilattice congruence on $S$ if and only if $C$ is convex.*

**Proof** Let $C$ be a congruence class of a regular semilattice congruence $\sigma$. By Theorem 2, we have $C = A \cap B$, where $A$ a semiprime ideal of $S$, and $B$ is a filter. If $a \leq x \leq b$, $a, b \in C, x \in S$. Obviously, $x \in C$.

Conversely, Let $C$ be convex. Let

$$A = \bigcup_{a \in C} (a], \quad B = \bigcup_{b \in C} [b).$$

Since $S$ is semilattice, it is easy to verify that $A$ is a semiprime ideal of $S$, and $B$ is a filter of $S$. Clearly, $C \subseteq A \cap B$. If $x \in A \cap B$, then there exist $a, b \in C$ such that $a \leq x \leq b$. Since $C$ is convex, we have $x \in C$. Thus $C = A \cap B$. By Theorem 2, $C$ is a congruence class of some regular congruence on $S$. □

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# 序半群的正则同余类

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**摘 要**: 本文的目的是给出一个序半群的子集能成为某个正则同余的同余类的刻画，同时我们可以容易看出 [5] 中的关于一般半群 (没有序关系) 的相应的结论仅是本文的结论的应用.