Journal of Mathematical Research & Exposition
Vol.21, No.2, 212–214, May, 2001

Subalgebras of Nilpotent Matrices *

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Abstract: In this note we describe explicitly the subalgebras of nilpotent matrices and obtain some interesting results.

Key words: nilpotent matrix; loop; subalgebra.

Classification: AMS(1991) 15A/CLC O151.21

Document code: A **Article ID:** 1000-341X(2001)02-0212-03

Linear spaces of nilpotent matrices have already been studied by several authors (e.g.[1],[2],[3]). In this paper we give a detailed description of an interesting class of subalgebras of nilpotent matrices.

Let K be an arbitrary field,n a positive integer and $M(n,k)$ the vector space of all $n \times n$-matrices over K. For $1 \leq i,j \leq n$, let E_{ij} be the matrix having coefficient 1 in the $i-$th row and $j-$th column and 0 elsewhere Then

$$M(n,k) = \bigoplus_{1 \leq i,j \leq n} KE_{ij}.$$

Let T be a subset of $\{(i,j) \mid 1 \leq i,j \leq n\}$ and $V_T := \bigoplus_{(i,j)\in T} KE_{ij}$.Let R_T be the subalgebra generated by $\{E_{ij},(i,j) \in T\}$. If $(i_1,i_2),(i_2,i_3),\ldots,(i_m,i_1) \in T(m \geq 1)$, we say that there is a loop in T.

Proposition 1 *The following conditions are equivalent:*

(1) Every matrix in V_T is nilpotent.

(2) There is no loop in T.

Proof (1) $\Longrightarrow$ (2) Suppose that T has a loop,then there are $i_1,i_2,\ldots,i_m$ such that $E_{i_1i_2},E_{i_2i_3},\ldots,E_{i_mi_1} \in V_T$.By (1) the matrix

$$A = E_{i_1i_2} + E_{i_2i_3} + \ldots + E_{i_mi_1} \in V$$

is nilpotent.But $A^m = E_{i_1i_2} + E_{i_2i_3} + \ldots + E_{i_mi_1} = A$,contradication.

*Received date: 1998-05-13
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(2) $\Longrightarrow$ (1) Suppose that $T=\{(i_1,j_1),(i_2,j_2),\ldots,(i_t,j_t)\}$.$\forall A\in V$, we have

$$A=k_1E_{i_1j_1}+k_2E_{i_2j_2}+\ldots+E_{i_tj_t},k_1,k_2,\ldots,k_t\in K.$$

$$A^{t+1}=\sum P_{k_1l_1k_2l_2\ldots k_{t+1}l_{t+1}}E_{k_1l_1}E_{k_2l_2}\ldots E_{k_{t+1}l_{t+1}},$$

$P_{k_1l_1k_2l_2\ldots k_{t+1}l_{t+1}}\in K$. Here $(k_1,l_1),(k_2,l_2),\ldots,(k_{t+1},l_{t+1})\in T$. Since$\#(T)=t$, so there exist $1\le i<j\le t+1$ such that $l_i=l_j$. If $E_{k_1l_1}E_{k_2l_2}\ldots E_{k_{t+1}l_{t+1}}\ne 0$, then

$$l_1=k_2,l_2=k_3,\ldots,l_i=k_{i+1},\ldots,l_t=k_{t+1}.$$

By the above discussion we have $i<j,l_i=k_{i+1}=l_j$. So $(k_{i+1},l_{i+1}),(k_{i+2},l_{i+2}),\ldots,(k_j,l_j)$ is a loop in T.Since T there is no loop,so $A^{t+1}=0$.Therefore,every matrix in V is nilpotent.

Let $A=(A_{ij})_{n\times n}$ be a matrix in $M(n,k)$. Define: $T_A=\{(i,j)\mid A_{ij}\ne 0\}$

Deduction *If there is no loop in T_A, then A is nilpotent.*

Proposition 2 *If there is no loop in T_A, then T_{A^2} there is no loop.*

Proof Since $A=\sum\limits_{(i,j)\in T_A}A_{ij}E_{ij}$, if T_{A^2} there is a loop $(k_1,l_1),(l_1,l_2),\ldots,(l_q,k_1)$,by $E_{k_il_i}=E_{k_ip_i}E_{p_il_i}$,then T_A there is a loop

$$(k_1,p_1),(p_1,l_1),(l_1,p_2),(p_2,l_2),\ldots,(l_q,p_q),(p_q,k_1).$$

Contradiction.

Deduction If there is no loop in T_A, then $T_{A^{2^s}}$ there is no loop,for any $s>0$.

For $T=\{(i_1,j_1),(i_2,j_2),\ldots,(i_t,j_t)\}$, define

$$C_T=E_{i_1j_1}+E_{i_2j_2}+\ldots+E_{i_tj_t}.$$

Proposition 3 *For any $A_1,A_2\in V_T,T_{(A_1\cdot A_2)}\subseteq T_{C_T^2}$.*

Generally, for any $A_1,A_2,\ldots,A_s\in V_T,T_{(A_1\cdot A_2\cdots A_s)}\subseteq T_{C_T^s}$.

Proposition 4 *If there is no loop in T, then $A_1\cdot A_2\cdots A_{2^s}$ is nilpotent for any $A_1,A_2,\ldots,A_{2^s}$ V_T.*

Proof Since there is no loop in T, so there is no loop in $T=T_{C_T}$, there is no loop in $T_{C_T^2}$, there is no loop in $T_{(A_1\cdot A_2\cdots A_{2^s})}$, $A_1\cdot A_2\cdots A_{2^s}$ is nilpotent.

Proposition 5 *If there is no loop in T, then*

$$A_1\cdot A_2\cdots A_{t+1}=0$$

for any $A_1,A_2,\cdots,A_{t+1}\in V_T,t=\#(T)$.

Proof By proof of proposition 1,$C_T^{t+1}=0$.Since $T_{A_1\cdot A_2\cdots A_{t+1}}\subseteq T_{C_T^{t+1}}=\emptyset$, so

$$A_1\cdot A_2\cdots A_{t+1}=0.$$

Proposition 6 *R_T is a nilpotent subalgebra if and only if there is no loop in T.*

Proof If R_T is nilpotent,then V_T is nilpotent,by proposition 1, there is no loop in T.

If there no loop in T. $\forall A_l \in R_T$, we have

$$A_l = \overrightarrow{A}^l_{11} \cdot \overrightarrow{A}^l_{12} \cdots \overrightarrow{A}^l_{1r_1} + \overrightarrow{A}^l_{21} \cdot \overrightarrow{A}^l_{22} \cdots \overrightarrow{A}^l_{2r_2} + \cdots + \overrightarrow{A}^l_{s1} \cdot \overrightarrow{A}^l_{s2} \cdots \overrightarrow{A}^l_{sr_s}.$$

Here $\overrightarrow{A}^l_{ij} \in V_T, i = 1, 2, \cdots, s, 1 \leq j \leq \max\{r_1, r_2, \cdots, r_s\}$.

By proposition 5,$A_1 A_2 \ldots A_{t+1} = 0$, here $t = \sharp(T)$. So R_T is nilpotent.

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幂 零 矩 阵 子 代 数

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摘 要: 本文较详细地描述了幂零矩阵子代数，并且获得了一些有趣的结果.