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Some Strange Identities Related to Faa di Bruno Formula *

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Abstract: In [1] various strange identities involving summations over partitions were proposed by using Faa di Bruno formula. In this paper, some other strange identities are got in an analogous way and a corollary is obtained by using inversion formulas of Lagrange.

Key words: identity; Faa di Bruno formula; partition.

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Proposition 1 *Let $f(t)$ and $g(t)$ be two infinitely differentiable functions with $f\circ g = g\circ f$. If we put $f_n = \frac{\mathrm{d}^n f}{\mathrm{d}t^n}\mid_{t=a}$, $g_m = \frac{\mathrm{d}^m g}{\mathrm{d}t^m}\mid_{t=a}$, $f_k^{(a)} = \frac{\mathrm{d}^k f}{\mathrm{d}t^k}\mid_{t=g(a)}$, $g_k^{(a)} = \frac{\mathrm{d}^k g}{\mathrm{d}t^k}\mid_{t=f(a)}$, then we have, for $n\geq 1$,*

$$\sum_{\sigma(n)}\frac{1}{k_1!k_2!\cdots k_n!}f_k^{(a)}\left(\frac{g_1}{1!}\right)^{k_1}\left(\frac{g_2}{2!}\right)^{k_2}\cdots\left(\frac{g_n}{n!}\right)^{k_n}$$
$$=\sum_{\sigma(n)}\frac{1}{k_1!k_2!\cdots k_n!}g_k^{(a)}\left(\frac{f_1}{1!}\right)^{k_1}\left(\frac{f_2}{2!}\right)^{k_2}\cdots\left(\frac{f_n}{n!}\right)^{k_n}, \tag{1}$$

where $\sigma(n)$ denote the set of partitions of n (n is positive integer), represented by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1+2k_2+\cdots+nk_n=n$, integers $k_i\geq 0$, $k=k_1+k_2+\cdots+k_n$.

Proof This follows immediately by applying Theorem C(p.139) in [2]. □

Example 1 Let $f(t)=e^t$, $g(t)=\ln(t)$, and $f\circ g=g\circ f$, then we obtain the following identity for $n\geq 1$ by using (1):

$$\sum_{\sigma(n)}\frac{1}{k_1!k_2!\cdots k_n!}\left(\frac{1}{1!}\right)^{k_1}\left(\frac{-1}{2!}\right)^{k_2}\cdots\left(\frac{(-1)^{n-1}(n-1)!}{n!}\right)^{k_n}$$
$$=\sum_{\sigma(n)}\frac{1}{k_1!k_2!\cdots k_n!}\frac{(-1)^{k-1}(k-1)!}{e^k}\left(\frac{e}{1!}\right)^{k_1}\left(\frac{e}{2!}\right)^{k_2}\cdots\left(\frac{e}{n!}\right)^{k_n}.$$

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After simplifying, we get

$$\sum_{\sigma(n)}\frac{(-1)^{k_2+2k_3+\cdots+(n-1)k_n}}{k_1!k_2!\cdots k_n!1^{k_1}2^{k_2}\cdots n^{k_n}}=\sum_{\sigma(n)}\frac{(-1)^{k-1}(k-1)!}{k_1!k_2!\cdots k_n!(1!)^{k_1}(2!)^{k_2}\cdots(n!)^{k_n}}.\ \square$$

Proposition 2 *Let $f(t),\varphi(t)$ be two infinitely differentiable functions and $f^{-1}(t)$ be the compositional inverse function of $f(t)$, $f\circ f^{-1}=t$. We denote*

$$\varphi(t)=\sum_{n\geq 0}\left[\begin{array}{c}\varphi\\ n\end{array}\right]t^n,$$

$$\varphi\circ f=\sum_{n\geq 0}\left[\begin{array}{c}\varphi\circ f\\ n\end{array}\right]t^n,$$

$$f^{-1}(t)=\sum_{n\geq 0}\left[\begin{array}{c}f^{-1}\\ n\end{array}\right]t^n,$$

$$(\varphi\circ f)_0^{(k)}=\frac{\mathrm{d}^k(\varphi\circ f)}{\mathrm{d}t^k}\Big|_{t=0},\left[\begin{array}{c}f^{-1}\\ k\end{array}\right]=\frac{\mathrm{d}^k f^{-1}(t)}{\mathrm{d}t^k}\Big|_{t=f(0)}.$$

Then we have the following formula

$$\left[\begin{array}{c}\varphi\\ n\end{array}\right]=\sum_{\sigma(n)}(\varphi\circ f)_0^{(k)}\prod_{i=1}^{n}\frac{1}{k_i!}\left[\begin{array}{c}f^{-1}\\ i\end{array}\right]^{k_i}. \tag{2}$$

Proof Notice that $\varphi(t)=\varphi\circ(f\circ f^{-1})(t)=(\varphi\circ f)\circ f^{-1}(t)$, and we can get (2) immediately by just applying Faa di Bruan formula:

$$\begin{aligned}\left[\begin{array}{c}\varphi\\ n\end{array}\right] &= \frac{1}{n!}\sum_{k=1}^{n}(\varphi\circ f)_0^{(k)}B_{n,k}(f_1^{-1},f_2^{-1},\cdots)\\ &\overset{*}{=} \frac{1}{n!}\sum_{k=1}^{n}(\varphi\circ f)_0^{(k)}\sum\frac{n!}{k_1!k_2!\cdots k_n!}\left(\frac{f_1^{-1}}{1!}\right)^{k_1}\left(\frac{f_2^{-1}}{2!}\right)^{k_2}\cdots\left(\frac{f_n^{-1}}{n!}\right)^{k_n}\\ &= \sum_{\sigma(n)}(\varphi\circ f)_0^{(k)}\prod_{i=1}^{n}\frac{1}{(k_i)!}\left[\begin{array}{c}f^{-1}\\ i\end{array}\right]^{k_i},\end{aligned}$$

where the $B_{n,k}$ are the exponential Bell polynomials and the second summation in (*) takes place over all integers $k_1\geq 0,k_2\geq 0,\cdots$, such that $k_1+2k_2+\cdots=n$, $k_1+k_2+\cdots=k$.

This completes the proof. □

Example 2 Let $\varphi(t)=\frac{1}{l!}x^l$, $f(t)=\ln(1+t), f^{-1}(t)=e^t-1$, then

$$\left[\begin{array}{c}\varphi\\ n\end{array}\right]=\begin{cases}\frac{1}{l!} & n=l\\ 0 & n\neq l\end{cases},\varphi\circ f=\frac{1}{l!}\ln^l(1+t)=\sum_{n\geq l}s(n,l)\frac{t^n}{n!},$$

$$(\varphi\circ f)_0^{(k)}=s(k,l),\left[\begin{array}{c}f^{-1}\\ i\end{array}\right]=\frac{1}{i!}\frac{\mathrm{d}^i(e^t-1)}{\mathrm{d}t^i}\Big|_{t=0}=\frac{1}{i!},$$

and we obtain the following identity involving the Stirling number of the first kind:

$$\sum_{\sigma(n)} s(k,l)\prod_{i=1}^{n}\frac{1}{k_i!(i!)^{k_i}} = \begin{cases} \frac{1}{l!}, & n=l, \\ 0, & n\neq l. \end{cases} \tag{3}$$

Also, we get another identity involving the Stirling number of the second kind in same way

$$\sum_{\sigma(n)} S(k,l)\prod_{i=1}^{n}\frac{(-1)^{i-1}}{k_i!i} = \begin{cases} \frac{1}{l!}, & n=l, \\ 0, & n\neq l. \end{cases} \tag{4}$$

And we have from (3) and (4)

$$\sum_{\sigma(n)} s(k,l)\prod_{i=1}^{n}\frac{1}{k_i!(i!)^{k_i}} = \sum_{\sigma(n)} S(k,l)\prod_{i=1}^{n}\frac{(-1)^{i-1}}{k_i!i}. \tag{5}$$

Example 3 Let $\varphi(t)=(\frac{2t}{t^2+1})^m$, $f(t)=e^t$, $f^{-1}(t)=\ln t$, then

$$\varphi\circ f=(\frac{2e^t}{e^{2t}+1})^m=\sum_{n\geq 0}\frac{E_n^{(m)}}{n!}t^n,$$

$$(\varphi\circ f)_0^{(k)}=E_n^{(m)},\ \begin{bmatrix} f^{-1} \\ i \end{bmatrix}=\frac{1}{i!}\frac{\mathrm{d}^i\ln t}{\mathrm{d}t^i}\Big|_{t=1}=\frac{(-1)^{i-1}}{i},$$

where $E_n^{(m)}$ denote the generalized Euler number, and we obtain the following identity

$$\sum_{\sigma(n)} E_n^{(m)}\prod_{i=1}^{n}\frac{1}{k_i!}\frac{(-1)^{(i-1)k_i}}{i^{k_i}} = \begin{cases} 0, & 0\leq n<m, \\ (-1)^{\frac{m-n}{2}}2^m\binom{\frac{3m-n}{2}-1}{m-1}, & n-m \text{ even},\ n\geq m, \\ 0, & n-m \text{ odd},\ n\geq m. \end{cases} \qquad \square$$

We can obtain the following corollary by using inversion formulas of Lagrange.

Corollary *Notations and hypotheses as in Proposition 2, and $f_0=0$, $f_1\neq 0$. Then we have*

$$\begin{bmatrix} \varphi \\ n \end{bmatrix}=\sum_{\sigma(n)}(\varphi\circ f)_0^{(k)}\prod_{i=1}^{n}\frac{1}{i(k_i)!}\begin{bmatrix} (\frac{f(t)}{t})^i \\ i-1 \end{bmatrix}^{k_i}, \tag{6}$$

$$\begin{bmatrix} \varphi \\ n \end{bmatrix}=\sum_{\sigma(n)}(k-1)!\begin{bmatrix} \varphi'(t)(\frac{f(t)}{t})^{-k} \\ k-1 \end{bmatrix}\prod_{i=1}^{n}\frac{1}{(k_i)!}\begin{bmatrix} f^{-1} \\ i \end{bmatrix}^{k_i}, \tag{7}$$

$$\begin{bmatrix} \varphi \\ n \end{bmatrix}=\sum_{\sigma(n)}(k-1)!\begin{bmatrix} (\frac{f(t)}{t})^{-k} \\ k-1 \end{bmatrix}\prod_{i=1}^{n}\frac{1}{(k_i)!}\begin{bmatrix} f\circ\varphi \\ i \end{bmatrix}^{k_i}. \tag{8}$$

Proof Formulas (6) and (7) can be got immediately by using Theorm A (p.148) and Theorem B (p.149) in [2], proposition 2, and inversion formulas of Lagrange. Formula (8) can be obtained by applying proposition (p.161) in [1] and inversion formulas of Lagrange. □

Example 4 Let $\varphi(t) = (1-t)^{-\alpha} = \sum_{n\geq 0}\binom{\alpha+n-1}{n}t^n$, $f(t) = 2tx - t^2$, then

$$\begin{bmatrix}\varphi\\ n\end{bmatrix} = \binom{\alpha+n-1}{n},\ \varphi\circ f = (1-2tx+t^2)^{-\alpha} = \sum_{n\geq 0} C_n^{(\alpha)}(x)t^n,\ (\varphi\circ f)_0^{(k)} = k!C_k^{(\alpha)}(x),$$

and

$$\left(\frac{f(t)}{t}\right)^{-i} = (2x-t)^{-i} = \sum_{l\geq 0}\binom{-i}{l}(2x)^{-i-l}(-1)^l t^l = \sum_{l\geq 0}\binom{l+i-1}{l}(2x)^{-i-l}t^l,$$

$$\begin{bmatrix}\left(\frac{f(t)}{t}\right)^{-i}\\ i-1\end{bmatrix} = \binom{2i-2}{i-1}(2x)^{-2i+1}.$$

Hence we get the following identity by using (6)

$$\frac{1}{2x}\binom{\alpha+n-1}{n} = \sum_{\sigma(n)} k!C_k^{(\alpha)}(x)\prod_{i=1}^{n}\frac{1}{i(k_i)!(2x)^{2i}}\binom{2(i-1)}{i-1},$$

where $x \neq 0$ and $C_k^{(\alpha)}(x)$ are the Gegenbauer polynomials. □

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与 Faa di Bruno 公式相关的一些奇异恒等式

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摘 要: 文 [1] 用 Faa di Bruno 公式找到了一些关于分拆集上求和奇异的恒等式，本文利用类似的方法找到了另外的一些奇异恒等式，并且利用 Lagrange 反演公式得到一个推论。