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New Auto-Darboux Transformation and Explicit Analytic Solutions for the Generalized KdV Equation with External Force Term *

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Abstract: In this paper, based on Lax pair of Riccati form of the generalized KdV(GKdV) equation with external force term, a new auto-Darboux transformation (ADT) is derived. As the application of the ADT, only if integration is needed, a series of explicit analytic solutions can be obtained, which contain solitary-like wave solutions. This method may be important for seeking more new and physical signficant analytic solutions of nonlinear evolution equations.

Key words: GKdV equation; Lax pair; Darboux transformation; analytic solution; solitary wave solution.

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1. Introduction

As is well known, Darboux transformation, which transforms one solution of an equation into anther solution of this eqaution, is of an important role in soliton theory[1-3,5,6]. Recently, Tian et al.[4] presented an approach, which was obtained from systems of Lax equations, applied to seek exact solutions of nonlinear evolution equations. But they only considered some simple and contant coefficient nonlinear eqautions. And we extented the method to derive Darboux transformation of the variable coefficient KdV equations[5].

For the generalized KdV equation with external force term[6]

$$u_t + h(u_{xxx} + 6uu_x) + 6fhu = g(t) + x(12hf^2 + f_t). \tag{1}$$

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where $h=h(t), g(t), f=f(t)$ are all arbitrary functions w.r.t.t. Many well-known nonlinear wave equations, such as KdV equation with externtion force term (f=0)

$$u_t + h(u_{xxx} + 6uu_x) = g(t). \tag{2}$$

and cylindrical KdV equation externtion force term ($f=\frac{1}{12t}, h=1$)

$$u_t + u_{xxx} + 6uu_x + \frac{1}{2t}u = g(t). \tag{3}$$

are both special cases of Eq.(1). only one soliton-like solution of Eq.(1) was obtained by using a direct method[6].

In this paper, we would like to give a new auto-Darboux transformation for the generalized KdV equation with the external force term (1) via the Lax pair of Riccati form. As examples to illustrate the transformation , some analytic solutions for Eq.(1) are obtained, which contain solitary-like wave solutions. When using the ADT, only integration is needed. Finally, we give some conclusions and problems which need to be studied further.

2. New auto-darboux transformation for Eq.(1)

For the Eq.(1) given, by use of WTC method[7] we easily show that Eq.(1) pass the Painleve test and there exsit the following conclusions

Proposition 1 *the Lax pairs for Eq.(1) can be written as*

$$\varphi_{xx} = (\lambda - u + fx + k)\varphi, \tag{4a}$$

$$\varphi_t = h(u_x - f)\varphi - h[2(u+2\lambda) + 4(fx+k)]\varphi_x, \tag{4b}$$

with $\lambda_t + 12hf\lambda = 0$, *i.e.*,

$$\lambda(t) = \lambda_0 \exp(-\int 12hf\mathrm{d}t). \tag{4c}$$

$$k = k(t) = \exp(-12\int hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t) + c]. \tag{4d}$$

where λ_0, c *are both arbitary constants.*

In order to use the Lax pairs (4a)-(4d) further, we rewrite the above form. Introducing the following transformation

$$\omega = \frac{\partial}{\partial x}\ln\varphi = \frac{\varphi_x}{\varphi}. \tag{5}$$

and substituting Eq.(5) into Eqs.(4a) and (4b), yield the following Lax pairs of Riccati form

$$\omega_x = \lambda + fx + k - u - \omega^2, \tag{6a}$$

$$\omega_t = hu_{xx} - h(4f + 2u_x)\omega - h[4(fx+k) + 2(u+2\lambda)]\omega_x. \tag{6b}$$

It is shown that the compatibility condition $\omega_{xt} = \omega_{tx}$ of Eqs.(6a) and (6b) is Eq.(1). Hence, if u and ω are solutions of Eqs.(6a) and (6b), then u is a solution for Eq.(1). In order to seek for analytic solutions of Eq.(1), we only need consider the Lax pairs (6) of Riccati form for Eq.(1) here.

Proposition 2 *Let*

$$A_m(x,t)=\frac{\partial}{\partial t}\left(\int \omega_m \mathrm{d}x\right)+h[4(fx+k)+2(u_m+2\lambda)]\omega_m-h(u_{mx}+2f). \tag{7}$$

and

$$\begin{aligned}B_m(x,t)=&h[4(fx+k)+2(u_m+2\lambda)]\exp(-2\int \omega_m \mathrm{d}x)+\\&2A_m(x,t)\int \exp(-2\int \omega_m \mathrm{d}x)\mathrm{d}x+\frac{\partial}{\partial t}[\int \exp(-2\int \omega_m \mathrm{d}x)\mathrm{d}x].\end{aligned} \tag{8}$$

If (u_m, ω_m) satisfy Eqs.(6a) and (6b), then $A_m(x,t)$ and $B_m(x,t)$ are both functions of only t, namely,

$$A_{mx}(x,t)=0 \ \ (i.e., A(x,t)=A(t)), \qquad B_{mx}(x,t)=0 \ \ (i.e., B(x,t)=B(t)). \tag{9}$$

Proposition 3 *Taking the following transformations*

$$u_{m+1}=2\frac{\partial^2}{\partial x^2}[\ln M_m(x,t)]+u_m+2\omega_{mx}, \tag{10a}$$

$$\omega_{m+1}=-\frac{\partial}{\partial x}[\ln M_m(x,t)]-\omega_m. \ \ (m=1,2,3,\ldots) \tag{10b}$$

with

$$\begin{aligned}M_m(x,t)=&\int \exp(-2\int \omega_m \mathrm{d}x)\mathrm{d}x+\exp[-2\int A_m(t)\mathrm{d}t]\times\\&[M_0-\int B_m(t)\exp(2\int A_m(t)\mathrm{d}t)\mathrm{d}t].\end{aligned} \tag{11}$$

Where M_0 is an arbitrary contant. If u_m and ω_m satisfy Eqs.(6a) and (6b), then u_{m+1} and ω_{m+1} also satisfy Eqs.(6a) and (6b).

Proof It is only needed to prove that u_{m+1} and ω_{m+1} also satisfy Eqs.(6a) and (6b), namely

$$\omega_{m+1,x}=\lambda+fx+k-u_{m+1}-\omega_{m+1}^2, \tag{12a}$$

$$\begin{aligned}\omega_{m+1,t}=&hu_{m+1,xx}-h(4f+2u_{m+1,x})\omega_{m+1}-\\&h[4(fx+k)+2(u_{m+1}+2\lambda)]\omega_{m+1,x}.\end{aligned} \tag{12b}$$

Substituting Eqs.(10a),(10b) and (11) into Eqs.(12a) and (12b) and combining Eqs.(7) and (8), it was easy to prove that Eqs.(12a) and (12b) hold.

According to Proposition 3, we get a new Darboux transformation for Eq.(1), namely, Eqs.(10a) and (10b) with u_m, ω_m satifying Eqs.(6a)and (6b) and λ, k, M_m satifying Eqs.(4c), (4d) and (11) respectively.

3. Explicit analytic solutions by using the ADT

For a solution (u_m, ω_m) of Eqs.(6), only integration is needed, then via the ADT (10a) and (10b), we can get another explicit analytic solution (u_{m+1}, ω_{m+1}) of Eq.(6). According to the same procedure, we can also obtain the third solution (u_{m+2}, ω_{m+2}) of Eq.(6), and so on. Thus $u_m, u_{m+1}, u_{m+2}, \ldots$ of these conclusions are exact solutions for Eq.(1). For instance

Case A Taking

$$\begin{aligned} u_1 &= \lambda + xf + k \\ &= xf + \exp(-\int 12hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c + \lambda_0], \quad \omega_1 = 0, \end{aligned} \tag{13}$$

it is clear that (u_1, ω_1) is a solution of Eq.(6), thus we get Therefore according to the definitions of $A_1(t), B_1(t)$ and $M_1(x,t)$ in Eqs.(7),(8) and (11), we have

$$A_1(t) = -3hf, \quad B_1(t) = 6h\exp(-\int 12hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c + \lambda_0], \tag{14}$$

$$\begin{aligned} M_1(x,t) =& x + \exp(6\int hf\mathrm{d}t)\{M_0 - \int 6h\exp(-\int 18hf\mathrm{d}t)\times \\ & [\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c + \lambda_0]\mathrm{d}t\}. \end{aligned} \tag{15}$$

Finally, according the ADT (10a) and (10b), we obtain

$$\begin{aligned} u_2 =& xf + \exp(-\int 12hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t) + c + \lambda_0] - \\ & 2\{x + \exp(6\int hf\mathrm{d}t)[M_0 - \int 6h\exp(- \\ & \int 18hf\mathrm{d}t)(\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c + \lambda_0)]\mathrm{d}t\}^{-2}. \end{aligned} \tag{16}$$

$$\begin{aligned} \omega_2 =& -\{x + \exp(6\int hf\mathrm{d}t)[M_0 - \int 6h\exp(-\int 18hf\mathrm{d}t)\times \\ & (\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c + \lambda_0)]\mathrm{d}t\}^{-1}. \end{aligned} \tag{17}$$

It is clear that u_2 is a rational fraction solution for Eq.(1). From the solution (u_2, ω_2) of Eq.(6) again, by virtue of the Auto-Darboux transformation(10), we can also derive another exact solution of Eq.(1). But the formal solution is rather complicated, we omit it here.

Case B Taking another solution of Eq.(6), namely

$$u_1(x,t) = xf + k = xf + \exp(-12\int hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c],$$

$$\omega_1 = \sqrt{\lambda_0}\exp(-\int 6hf\mathrm{d}t). \tag{18}$$

Then by use of Eq.(18), we have

$$A_1(t) = h\exp(-\int 18hf\mathrm{d}t)[2\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + 2c + 4\lambda_0] - 3hf, B_1(t) = 0. \tag{19}$$

Substituting $A_1(t), B_1(t)$ into $M_1(x,t)$ yields

$$\begin{aligned} M_1(x,t) = & -\frac{1}{2\sqrt{\lambda_0}}\exp[-2\sqrt{\lambda_0}xe^{-\int 6hf\mathrm{d}t} + \int 6hf\mathrm{d}t] + M_0\exp[-2h\exp(- \\ & \int 18hf\mathrm{d}t)(2\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + 2c + 4\lambda_0) - 6hf], \end{aligned} \tag{20}$$

Finally, via use of auto-Darboux transformation (10), we can obtain

$$u_2(x,t) = \frac{2(M_{1xx}M_1 - M_{1x}^2)}{M_1^2} + xf + \exp(-12\int hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c], \tag{21}$$

$$\omega_2(x,t) = -\frac{M_{1x}}{M_1} - \sqrt{\lambda_0}\exp(-\int 6hf\mathrm{d}t). \tag{22}$$

In fact, $u_2(x,t)$ is a soliton-like solution for Eq.(1). which can be rewritten as follows.

(Bi) As $M_0 < 0$, we get a bell-type soliton-like solution for Eq.(1),

$$\begin{aligned} u_{21}(x,t) = & \lambda_0\exp(-12\int fh\mathrm{d}t)sech^2\{-\sqrt{\lambda_0}xe^{-\int 6hf\mathrm{d}t} + \int 3hf\mathrm{d}t + h\exp(-\int 18hf\mathrm{d}t)\times \\ & [2\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + 2c + 4\lambda_0] - 3hf + \frac{1}{2}\ln(-\frac{1}{2M_0\sqrt{\lambda_0}})\} + xf + \\ & \exp(-12\int hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)dt + c], \end{aligned} \tag{23}$$

(Bii) When $M_0 > 0$, we can derive another type singular soliton-like solution for Eq.(1),

$$\begin{aligned} u_{22}(x,t) = & \lambda_0\exp(-12\int fh\mathrm{d}t)csch^2\{-\sqrt{\lambda_0}xe^{-\int 6hf\mathrm{d}t} + \\ & \int 3hf\mathrm{d}t + h\exp(-\int 18hf\mathrm{d}t)[2\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + \\ & 2c + 4\lambda_0] - 3hf + \frac{1}{2}\ln(\frac{1}{2M_0\sqrt{\lambda_0}})\} + xf + \\ & \exp(-12\int hf\mathrm{d}t)[\int g\exp(12\int hf\mathrm{d}t)\mathrm{d}t + c], \end{aligned} \tag{24}$$

4. Summary and conclusions

In summary, we derive a new auto-Darboux transformation(ADT) for the generalized KdV (GKdV) equation with external force term, using the Lax pair of the GKdV equation.

And based on the ADT, three types of exact analytic solutions are obtained, Solitary-like wave solutions of which may be important for explaining some physical phenomena. This method may be also extented to systems of nonlinear evolution equations and higher dimensional nonlinear wave equations. In addition, Lax pair may be applied to find other properities, such as symmetries, conservation laws, and so on. These problems need to be studied further.

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含外力项的广义 KdV 方程的新自 Darboux 变换和显式解析解

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摘 要: 基于 Riccati 形式的 Lax 对，本文推得了含外力项的广义 KdV 方程的新自 Darboux 变换. 当应用这个变换时，仅需要做积分，就能获得一系列显示解析解，其中包含类孤波解. 这种途径对于寻找非线性发展方程新的具有物理意义的解或许是有用的.