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The Asymptotic Behavior of a Class of Second Order Differential Equation *

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Abstract: In the paper the asympotic behavior of the solution of the following second order differential equation

$$y'' + q(t)f(y) = 0$$

is investigated. Some results for asymptotic behavior of the nonoscillation solution are obtained.

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1. Introduction

Motivated by application, oscillation theory of second order differential equation have been extensively studied during the past decades. Papers [1–4] were concerned with the following differential equation

$$y'' + Q(t)|y|a(t)\text{sign}(y) = 0, \tag{1}$$

where $a(t) \in C[t_0, +\infty), t_0 > 0$ and $a(t) < 0$ for arbitrarity large values of t. Some ocillation criteria for Eq(1) are obtained.

In this paper, we are interested in the following differential equation in the form of

$$y'' + q(t)f(y) = 0, \tag{2}$$

where $q(t) \in C^1[t_0, +\infty), t_0 > 0, f(y)$ is a continuous real-valued function and satisfies $f(y)y > 0$ for all $y \neq 0$, and mainly study the asymptotic behavior of the nonoscillation of

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Eq(2). Throughout the paper we shall restrict our attention only to the solutions which exist in $[T_0,+\infty),(T_0>t_0)$ and satisfy $\mathrm{Sup}\{|y(t)|:t>T\}>0(T>T_0)$.

As a customary, a solution is said oscillatory if it has arbitrarily large zeros, and nonoscillatory if it is eventually positive or negative.

Theorem 1 *If $q(t)>0,q'(t)>0(t\geq t_0)$, and $\lim_{|y|\to\infty}\int_{y_0}^{y}f(u)\mathrm{d}u=+\infty$, then each nonoscillation solution of Eq(2) is bounded and $y(t)$ has a horizontal asymtote, i.e., there exists some constant L such that $\lim_{t\to+\infty}y(t)=L$.*

Proof For the nonoscillation solution $y(t)$ of Eq(2), there exists some $t_1>t_0$ such that $y(t)\neq 0$ when $t\geq t_1$. Therefore

$$[\frac{y'(t)}{y(t)}]'=-\frac{q(t)f(y)}{y}-[\frac{y'(t)}{y(t)}]^2\leq -q(t)\frac{f(y)}{y}<0. \tag{3}$$

It is easy to say that $\frac{y'(t)}{y(t)}$ is a monotonously decreasing function. Suppose $y(t)>0,t>t_1$, if there is $t_2>t_1$, such that $\frac{y'(t)}{y(t)}<0(t\geq t_2)$, then for any $t\geq t_2,y'(t)<0$. From $y''=-q(t)f(y(t))<0$, we get that $y(t)$ is a monotonously decreasing and convex function. Therefore, there must exist $t_3\geq t_2$ such that $y(t_3)<0$. Which is contridiction with our hypothesis. That is to say that $\frac{y'(t)}{y(t)}>0$ and $y'(t)>0$ for $t\geq t_1$. If $y(t)<0,t\geq t_1$, the proof is similar to the above, we also have $\frac{y'(t)}{y(t)}>0$ and $y'(t)<0$ for $t\geq t_1$.

From the above it implies that the nonoscillation solution $y(t)$ of Eq(2) is either monotonously increasing or decreasing when t is enough large.

Multiplying the both sides of Eq(2) by $y'(t)$, then integrating from t_0 to t, we have

$$\int_{t_0}^{t}y''(\tau)y'(\tau)\mathrm{d}\tau+\int_{t_0}^{t}q(\tau)f(y(\tau))y'(\tau)\mathrm{d}\tau=0.$$

That is

$$\frac{1}{2}[y'(t)]^2+\int_{t_0}^{t}q(\tau)f(y(\tau))y'(\tau)\mathrm{d}\tau=\frac{1}{2}[y'(t_0)]^2. \tag{4}$$

Denote $I(y)=\int_{y_0}^{y}f(u)\mathrm{d}u,y_0=y(t_0)$, from (4) we get

$$\frac{1}{2}[y'(t)]^2+q(t)I(y(t))=\frac{1}{2}[y'(t_0)]^2+q(t_0)I(y(t_0))+\int_{t_0}^{t}q'(\tau)I(y(\tau))\mathrm{d}\tau. \tag{5}$$

Thus, we have

$$q(t)I(y(t))\leq K+\int_{t_0}^{t}\frac{q'(\tau)}{q(\tau)}q(\tau)I(y(\tau))\mathrm{d}\tau.$$

Where $K=\frac{1}{2}[y'(t_0)]^2+q(t_0)I(y(t_0))$. Since $q'(t)>0$, applying Gronwall's inequality, we have

$$I(y(t))\leq\frac{K}{q(t_0)}<+\infty. \tag{6}$$

Inequality (6) implies that $y(t)$ must be bounded. And if $y(t)>0,y(t)y'(t)>0$, for enough large t, then there must exist $L>0$ such that $\lim_{t\to+\infty}y(t)=L$. Similarly, if $y(t) < 0, y(t)y'(t) > 0$, for enough large t, then there must exist $L < 0$ such that $\lim_{t\to+\infty} y(t) = L$. □

Theorem 2 *If $q(t) > 0, q'(t) > 0$ for $t \geq t_0$ and $\lim_{|y|\to+\infty} \int_{y_0}^{y} f(u)\mathrm{d}(u) = +\infty$, then $y'(t)$, the derivative of each nonscilation solution $y(t)$ of Eq(2) is bounded and $y'(t)$ has a horizontal asymptote, i.e., there exists some constant L such that $\lim_{t\to+\infty} y'(t) = L$.*

Proof If $y(t)$ is any one nonoscillation solution of Eq(2), from the proof of Theorem 1 there is $t_1 \geq t_0$, such that $y(t)y'(t) > 0$ when $t \geq t_1$. If $y(t) > 0$, with (2) we get

$$y''(t) = -q(t)f(y(t)) < 0, \quad t \geq t_1$$

This implies that $y'(t)$ is monotonously decreasing. If $y(t) < 0$, with (2) we get

$$y''(t) = -q(t)f(y(t)) > 0, \quad t \geq t_1$$

This implies that $y'(t)$ is monotonously increasing. And also

$$\int_{t_0}^{t} q'(\tau)I(y(\tau))\mathrm{d}\tau = q(t)I(y(t)) - q(t_0)I(y(t_0)) - \int_{t_0}^{t} q(\tau)f(y(\tau))y'(\tau)\mathrm{d}\tau. \qquad (7)$$

According to (5), we have

$$\frac{1}{2}[y'(t)]^2 \leq K - q(t_0)I(y(t_0)). \qquad (8)$$

This is to say that $|y'(t)|$ is bounded. With the monotonousness of $y'(t)$ we get if $y(t) > 0$, then there exists a constant $L \geq 0$ such that $\lim_{t\to+\infty} y'(t) = L$. And if $y(t) < 0$, then there exists a constant $L \leq 0$ such that $\lim_{t\to+\infty} y'(t) = L$. □

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