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Convergence Rates for Sums of Non-identically Distributed Random Elements *

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Abstract: Under some conditions on probability, we discuss the results in [1] for the part $r > 1$, which Yang[1] had not solved, such that the convergence rates are solved thoroughly in this case. Obviously, our conditions are weaker than Yang's corresponding moment conditions. Meanwhile, Banach spaces of type $p(1 < p \le 2)$ are characterized. For $0 < t < 1$, we prove that the corresponding results hold for independent random elements in any Banach space. As application we give the corresponding results for randomly indexed partial sums.

Key words: convergence rate; B-valued random element; partial sum; space of type p.

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1. Introduction

Let B be a real separable Banach space with norm $\|\cdot\|$ and $\{X_n, n \ge 1\}$ a sequence of B-valued independent random elements and put $S_n = \sum_{i=1}^n X_i, n \ge 1$.

Banach space B is called space of type p $(1 \le p \le 2)$ if for any zero mean B-valued independent random element sequence $\{X_n, n \ge 1\}$, there exists $C = C_p > 0$ such that

$$E\|\sum_{i=1}^n X_i\|^p \le C \sum_{i=1}^n E\|X_i\|^p, n \ge 1.$$

Let S be the class of positive non-decreasing function $\varphi(x)$ on $R^+ = [0, \infty)$ satisfying the following conditions:

(i) There exists a constant $k = k(\varphi) > 0$ such that

$$\varphi(xy) \le k(\varphi(x) + \varphi(y)), \quad \forall x, y \in R^+.$$

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(ii) $x/\varphi(x)$ is non-decreasing for sufficiently large x.

Yang[1] obtained the following result:

Theorem A *Let $0 < t < 2$, and let B be a Banach space of type 2. When $1 \le t < 2$, further let $EX_n = 0$. Suppose that $\varphi(x) \in S, \delta > 0, d = 1$ or -1. If*

$$\sum_{i=1}^{n} E[\|X_i\|^t(\varphi(\|X_i\|^t))^{-d}]^{1+\delta} = O(n),$$

then for every $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\le k\le n} \|S_k\| \ge \varepsilon \cdot (n(\varphi(n))^d)^{1/t}) < \infty,$$
$$\sum_{n=1}^{\infty} \frac{1}{n} P(\|S_n\| \ge \varepsilon \cdot (n(\varphi(n))^d)^{1/t}) < \infty.$$

In 1997, we improved Theorem A and obtained a characterization of Banach spaces of type $p(1 < p \le 2)$. In this paper, under some weaker conditions on probability than Yang's[1] corresponding moment conditions, we discuss the convergence rates corresponding Theorem A for the part $r > 1$, we solved thoroughly in this case, and also obtain the characterization of Banach spaces of type $p(1 < p \le 2)$. For $0 < t < 1$, we prove that the corresponding results hold for independent random elements in any Banach space. As an application we give the corresponding results for randomly indexed partial sums.

In the sequel, C and c denote positive finite constants whose value may vary from statements to statements. $l(x)$ is a slowly varying function as $x \to \infty$.

2. Main results

Theorem 2.1 *Let $1 \le t < 2, \varphi(x) \in S, r > 1, \delta > 0, d = 1$ or -1. The following statements are equivalent:*

(a) B is of type p for some $t < p \le 2$;

(b) For each sequence $\{X_n\}$ of zero mean B-valued independent random elements, if $\sum_{i=1}^n P(\|X_i\|^t(\varphi(\|X_i\|))^{-d} > x) \le Cnx^{-(r+\delta)}$ for sufficiently large x and n, then $\forall \varepsilon > 0$, we have

(i) $\sum_{n=1}^\infty n^{r-2} l(n) P(\|S_n\| \ge \varepsilon \cdot (n(\varphi(n))^d)^{1/t}) < \infty$;

(ii) $\sum_{n=1}^\infty n^{r-2} l(n) P(\max_{1\le k\le n} \|S_k\| \ge \varepsilon \cdot (n(\varphi(n))^d)^{1/t}) < \infty$;

(iii) $\sum_{n=1}^\infty n^{r-2} l(n) P(\sup_{k\ge n} (\|S_k\|/(k(\varphi(k))^d)^{1/t}) \ge \varepsilon) < \infty$.

Theorem 2.2 *Let $0 < t < 1, \varphi(x) \in S, r > 1, \delta > 0$.*

(a) If $\sum_{i=1}^n P(\|X_i\|^t/\varphi(\|X_i\|) \ge x) \le Cnx^{-(r+\delta)}$ for sufficiently large x and n, then we have the following statements, which are also equivalent:

(i) $\sum_{n=1}^\infty n^{r-2} l(n) P(\|S_n\| \ge \varepsilon \cdot (n\varphi(n))^{1/t}) < \infty, \quad \forall \varepsilon > 0$.

(ii) $\sum_{n=1}^\infty n^{r-2} l(n) P(\max_{1\le k\le n} \|S_k\| \ge \varepsilon \cdot (n\varphi(n))^{1/t}) < \infty, \quad \forall \varepsilon > 0$.

(iii) $\sum_{n=1}^\infty n^{r-2} l(n) P(\sup_{k\ge n} (\|S_k\|/(k\varphi(k))^{1/t}) \ge \varepsilon) < \infty, \quad \forall \varepsilon > 0$.

(b) If $\sum_{i=1}^n P(\|X_i\|^t \varphi(\|X_i\|) \ge x) \le Cnx^{-(r+\delta)}$ for sufficiently large n and $x \ge 0$, then we have the following statements, which are also equivalent:

(i) $\sum_{n=1}^{\infty} n^{r-2}l(n)P(\|S_n\| \geq \varepsilon \cdot (n/\varphi(n))^{1/t}) < \infty, \quad \forall \varepsilon > 0.$
(ii) $\sum_{n=1}^{\infty} n^{r-2}l(n)P(\max_{1\leq k\leq n}\|S_k\| \geq \varepsilon \cdot (n/\varphi(n))^{1/t}) < \infty, \quad \forall \varepsilon > 0.$
(iii) $\sum_{n=1}^{\infty} n^{r-2}l(n)P(\sup_{k\geq n}(\|S_k\|/(k/\varphi(k))^{1/t}) \geq \varepsilon) < \infty, \quad \forall \varepsilon > 0.$

3. Proofs of main results

Lemma 3.1[2] *Let $\varphi(\cdot) \in S, \delta > 0$, then for any $x \geq 0$, there exists positive constant C such that*

$$C\varphi(x) \leq \varphi(x\varphi(x)) \leq C\varphi(x),\ C\varphi(x) \leq \varphi(x/\varphi(x)) \leq C\varphi(x),\ C\varphi(x) \leq \varphi(x^{\delta}) \leq C\varphi(x).$$

Lemma 3.2[3] *Let $\{X_n\}$ be a sequence of zero mean independent random elements in a Banach space of type p. Then for $q \geq p$,there exists $C > 0$ such that*

$$E\max_{1\leq k\leq n}\|S_k\|^q \leq (96q)^q\{(C\sum_{i=1}^{n} E\|X_i\|^p)^{q/p} + E\max_{1\leq k\leq n}\|X_k\|^q\}.$$

We prove only Theorem 2.1 for the case in $d = 1$, the proof of Theorem 2.1 for the case in $d = -1$ and Theorem 2.2 is analogous.

Proof of Theorem 2.1. (a) $\Rightarrow$(b) Clearly, (ii) and (iii) imply (i), so, we need only to prove (ii) and (iii). First, we prove (ii). Let $Y_{ni} = X_i I(\|X_i\| \leq (n\varphi(n))^{1/t})$, $S_{nk} = \sum_{i=1}^{k} Y_{ni}$, $T_{nk} = S_k - S_{nk}$. Obviously

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}l(n)P(\max_{1\leq k\leq n}\|S_k\| \geq \varepsilon \cdot (n\varphi(n))^{1/t}) \\
&\leq \sum_{n=1}^{\infty} n^{r-2}l(n)P(\max_{1\leq k\leq n}\|T_{nk}\| \geq \frac{\varepsilon}{2} \cdot (n\varphi(n))^{1/t}) + \\
&\quad \sum_{n=1}^{\infty} n^{r-2}l(n)P(\max_{1\leq k\leq n}\|S_{nk}\| \geq \frac{\varepsilon}{2} \cdot (n\varphi(n))^{1/t}) =: I_1 + I_2.
\end{aligned}$$

By the monotonicity of $x/\varphi(x)$ and Lemma 3.1 we have

$$I_1 \leq \sum_{n=1}^{\infty} n^{r-2}l(n)\sum_{k=1}^{n} P(\|X_k\| > (n\varphi(n))^{1/t}) \leq C\sum_{n=1}^{\infty} n^{-(1+\delta)}l(n) + C < \infty.$$

From the definition of $\varphi(x)$ and Lemma 3.1, we may obtain that for every $\alpha, \beta > 0$

$$\varphi(x^{\alpha}) \leq Cx^{\beta} \tag{3.1}$$

for sufficiently large x. From the definition of $\varphi(x)$, by $EX_n = 0$, (3.1) and Lemma 3.1, for sufficiently large n, we have

$$\max_{1\leq k\leq n}\|ES_{nk}\|/(n\varphi(n))^{1/t} \leq \sum_{i=1}^{n} E\|X_i\|I(\|X_i\| > (n\varphi(n))^{1/t})/(n\varphi(n))^{1/t} \to 0 \ \text{ as } \ n \to \infty.$$

Thus, to prove $I_2 < \infty$, it suffices to show that

$$I_2^* =: \sum_{n=1}^{\infty} n^{r-2} l(n) P(\max_{1\le k\le n} \|S_{nk} - ES_{nk}\| \ge \varepsilon \cdot (n\varphi(n))^{1/t}) < \infty, \quad \forall \varepsilon > 0.$$

In fact, by the Markov inequality, choose $q > \max\{p, \frac{pt(r-1)}{p-t}, \frac{p(r-1)}{r+\delta-1}, rt\}$, from Lemma 3.2 we get

$$\begin{aligned} I_2^* &\le C\sum_{n=1}^{\infty} n^{r-2} l(n) \cdot (n\varphi(n))^{-q/t} E(\max_{1\le k\le n} \|S_{nk} - ES_{nk}\|)^q \\ &\le C\sum_{n=1}^{\infty} n^{r-2} l(n) \cdot (n\varphi(n))^{-q/t} [\sum_{i=1}^{n} E\|Y_{ni}\|^p]^{q/p} + \\ &\quad C\sum_{n=1}^{\infty} n^{r-2} l(n) \cdot (n\varphi(n))^{-q/t} E \max_{1\le i\le n} \|Y_{ni}\|^q \\ &=: I_3 + I_4. \end{aligned}$$

By the monotonicity of $\varphi(x), x/\varphi(x)$ and Lemma 3.1 we get

$$\begin{aligned} I_3 &= C\sum_{n=1}^{\infty} n^{r-2} l(n) \cdot (n\varphi(n))^{-q/t} [\sum_{i=1}^{n} \int_0^{n\varphi(n)} x^{p/t-1} P(\|X_i\|^t > x) dx]^{q/p} \\ &\le C\sum_{n=1}^{\infty} n^{r-2-q/t+q/p} l(n) \cdot (\varphi(n))^{-q/t} + \\ &\quad \begin{cases} C\sum_{n=1}^{\infty} n^{r-2+q/p-q(r+\delta)/p} l(n) & \text{if } \frac{p}{t} - r - \delta > 0 \\ C\sum_{n=1}^{\infty} n^{r-2+q/p-q/t} l(n) \cdot (\varphi(n))^{q(r+\delta)/p-q/t} [\log(n\varphi(n))]^{q/p} & \text{if } \frac{p}{t} - r - \delta = 0 \\ C\sum_{n=1}^{\infty} n^{r-2+q/p-q/t} l(n) \cdot (\varphi(n))^{q(r+\delta)/p-q/t} & \text{if } \frac{p}{t} - r - \delta < 0 \end{cases} \\ &< \infty. \end{aligned}$$

Similarly, we can get $I_4 < \infty$.

Next, we prove (iii). By the property of $l(x)$, the definition of $\varphi(x)$ and Lemma 1 of Bai and Su[4] we have

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} l(n) P(\sup_{k\ge n} (\|S_k\|/(k\varphi(k))^{1/t}) \ge \varepsilon) \le \\ &\quad C\sum_{m=0}^{\infty} 2^{m(r-1)} l(2^m) P(\max_{2^m \le k < 2^{m+1}} \|S_k\| \ge \varepsilon \cdot (2^m \varphi(2^m))^{1/t}). \end{aligned}$$

Denote by $Y_{mi} = X_i I(\|X_i\| \le (2^m\varphi(2^m))^{1/t})$, $S_{mk} = \sum_{i=1}^{k} Y_{mi}$, $T_{mk} = S_k - S_{mk}$. As in the proof in (ii), we get

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} l(n) P(\sup_{k\ge n} (\|S_k\|/(k\varphi(k))^{1/t}) \ge \varepsilon) \le \\ &\quad C\sum_{m=0}^{\infty} 2^{m(r-1)} l(2^m) P(\max_{2^m \le k < 2^{m+1}} \|T_{mk}\| \ge \frac{\varepsilon}{2} \cdot (2^m \varphi(2^m))^{1/t}) + \\ &\quad C\sum_{m=0}^{\infty} 2^{m(r-1)} l(2^m) P(\max_{2^m \le k < 2^{m+1}} \|S_{mk}\| \ge \frac{\varepsilon}{2} \cdot (2^m \varphi(2^m))^{1/t}) < \infty. \end{aligned}$$

(b) $\Rightarrow$ (a) Since (ii) and (iii) imply (i), if (b) is satisfied, taking $l(x)=1, \varphi(x)=1$, and further let $\{X_n\}$ be symmetric, assume that $\sum_{i=1}^n P(\|X_i\|^t > x) \leq Cnx^{-(r+\delta)}$ for sufficiently large x, n. Then from (b) we have $\sum_{n=1}^\infty n^{r-2} P(\|S_n\| \geq \varepsilon \cdot n^{1/t}) < \infty, \forall \varepsilon > 0$. Hence, as in the proof of Proposition 1.1 in Liang et al.[5] we get

$$S_n / n^{1/t} \xrightarrow{P} 0. \tag{3.2}$$

Let $\{x_n, n \geq 1\}$ be any bounded sequence in B, and $\{\varepsilon_n, n \geq 1\}$ be the Bernoulli sequence of random variables. Set $X_n = \varepsilon_n x_n$, $n \geq 1$. Then $\{X_n, n \geq 1\}$ is a sequence of independent symmetric B-valued random elements.

Since $\{x_n\}$ is a bounded sequence in B, for all $n \geq 1$ and sufficiently large x we have $P(\|X_i\|^t > x) = 0$. Hence, for all $n \geq 1$ and sufficiently large x we have

$$\sum_{i=1}^n P(\|X_i\|^t > x) \leq Cnx^{-(r+\delta)}.$$

By (3.2) we get $\frac{1}{n^{1/t}} \sum_{i=1}^n \varepsilon_i x_i \xrightarrow{P} 0$. From Proposition 5 in Wu and Wang[6,P.249], it follows that $\frac{1}{n^{1/t}} \sum_{i=1}^n \varepsilon_i x_i \to 0$ $a.s.$ which implies that B is of type p for $t < p \leq 2$.

4. Convergence rates for randomly indexed sums

In this section, let $\{\tau_n, n \geq 1\}$ be a sequence of non-negative, integer valued random variables, and τ be a positive random variable. All random variables are defined on the same probability space.

Theorem 4.1 *Let $1 \leq t < 2, r > 1, \varphi(x) \in S, \delta > 0, d = 1$ or -1. The following statements are equivalent:*

(a) B is of type p for some $t < p \leq 2$;

(b) Under the assumptions of Theorem 2.1(b), if there exists $\gamma > 0$ such that $\sum_{n=1}^\infty n^{r-2} l(n) P(\frac{\tau_n}{n} < \gamma) < \infty$, then $\forall \varepsilon > 0$ we have

$$\sum_{n=1}^\infty n^{r-2} l(n) P(\|S_{\tau_n}\| \geq \varepsilon \cdot (\tau_n (\varphi(\tau_n))^d)^{1/t}) < \infty.$$

(c) Under the assumptions of Theorem 2.1(b), if there exists $\varepsilon_0 > 0$ such that

$$\sum_{n=1}^\infty n^{r-2} l(n) P(|\frac{\tau_n}{n} - \tau| > \varepsilon_0) < \infty,$$

with $P(\tau \leq B) = 1$ for some $B > 0$, then $\forall \varepsilon > 0$ we have

$$\sum_{n=1}^\infty n^{r-2} l(n) P(\|S_{\tau_n}\| \geq \varepsilon \cdot (n (\varphi(n))^d)^{1/t}) < \infty.$$

Similarly, we can get the results on randomly indexed partial sums corresponding to Theorem 2.2.

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非同分布随机元和的收敛速度

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摘 要: 本文在弱于 [1] 的矩条件下，对 $r > 1$ 讨论了 $p(1 < p \le 2)$ 型空间中非同分布随机元和的收敛速度，同时获得了 p 型巴拿哈空间的刻划，作为应用，我们给出了随机元随机指标部分和的收敛速度.