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An Extension and a Correction Concerning Raney's Lemma *

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Abstract: We give an extension of Raney's lemma and correct a generalization of Raney's lemma in R.L.Graham et al's Concrete Mathematics.

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1. An extension of Raney's lemma

Consider a sequence $< a_1, a_2, \cdots, a_m >$ of real numbers with $\sum_{i=1}^{m} a_i > 0$.

We arrange $< a_1, a_2, \cdots, a_m >$ on a circle in clockwise direction, and let $(a_1, a_2, \cdots, a_m)$ denote this circle arrangement of length m. For given $a_i, i = 1, 2, \cdots, m$, define $a_{i_1} = a_i, a_{i_2} = a_{i+1}, \cdots, a_{i_m} = a_{i-1}$ with $a_j = a_k$ if $j \equiv k(\mathrm{mod} m)$. If $\sum_{j=1}^{k} a_{i_j} > 0$ for all $k, k = 1, 2, \cdots, m$, we call a_i an initial point of $(a_1, a_2, \cdots, a_m)$.

Now, we prove the existence of initial point in $(a_1, a_2, \cdots, a_m)$ by induction on m.

If $m = 1$, a_1 is an initial point.

For given $(a_1, a_2, \cdots, a_m, a_{m+1})$ of length $m + 1$, if $a_i \geq 0, i = 1, 2, \cdots, m + 1$, since $\sum_{i=1}^{m+1} a_i > 0$, if there exists $a_k > 0, a_k$ is an initial point; If there exists $a_i < 0$, we consider the following algorithm.

If $\langle a_{k_1}, a_{k_2}, \cdots, a_{k_l} \rangle$ satisfies

$$a_{k_1} a_{k_m} < 0, \quad a_{k_1} a_{k_{l+1}} < 0, \quad a_{k_1} a_{k_j} \geq 0$$

for all $j = 1, 2, \cdots, l$, then we ignore the sequence structure of $\langle a_{k_1}, a_{k_2}, \cdots, a_{k_l} \rangle$ while regarding it as a big point with value $\sum_{j=1}^{l} a_{k_j}$. The circle arrangement can be partitioned into the union of such subsequences. Denote these big points by $A_1, A_2, \cdots$, beginning from any chosen big point A_1. So, we obtain a circle arrangement $(A_1, A_2, \cdots)$ with $A_{i_1} A_{i_2} < 0$. Since

$$\sum A_i = \sum_{j=1}^{m+1} a_j > 0,$$

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there exist two consecutive big points A_{k_1}, A_{k_2} such that

$$A_{k_1} > 0, \quad A_{k_1} + A_{k_2} > 0.$$

Regarding $\langle A_{k_1}, A_{k_2}\rangle$ as a new big point B with value $A_{k_1} + A_{k_2}$ and replacing $\langle A_{k_1}, A_{k_2}\rangle$ by B in $(A_1, A_2, \cdots)$, we obtain a new circle arrangement of length $\le m$. There is an initial point in this new circle arrangement by induction assumption. Obviously, if $A_i \ne B$ is an initial point, then the first element of the subsequence expressed by A_i is an initial point of $(a_1, a_2, \cdots, a_{m+1})$; if B is an initial point, then the first element of the subsequence expressed by A_{k_1} is an initial point of $(a_1, a_2, \cdots, a_{m+1})$.

Summing up the above discussion, we obtain the following

Theorem 1 *There exists an initial point in circle arrangement* $(a_1, a_2, \cdots, a_m)$ *of real numbers with* $\sum_{i=1}^m a_i > 0$.

If $a_i's$ $(i = 1, 2, \cdots, m)$ are integers with $\sum_{i=1}^m a_i > 0$, and a_{k_1}, a_{k_s} $(s > 1)$ are two initial points in $(a_1, a_2, \cdots, a_m)$, then

$$\sum_{i=1}^m a_i = \sum_{j=1}^{s-1} a_{k_j} + \sum_{j=s}^m a_{k_j} \ge 1 + 1 = 2,$$

so,there exists only one initial point in $(a_1, a_2, \cdots, a_m)$ of integers with $\sum_{i=1}^m a_i = 1$, viz., exactly one of the cyclic shifts

$$\langle a_1, a_2, \cdots, a_m\rangle, \langle a_2, \cdots, a_m, a_1\rangle, \cdots, \langle a_m, a_1, \cdots, a_{m-1}\rangle$$

has all of its partial sums positive.This is the conclusion of Raney's lemma (see[1]). Hence, Theorem 1 can be regarded as an extension of Raney's lemma.

Remark We point out that Theorem 1 can be extented to the setting of ordered semi-group, the details omitted here.

2. A correction of a generalization of Raney's lemma

Consider circle arrangement $(a_1, a_2, \cdots, a_m)$ of integers with $a_i \le 1$ for all i, and $\sum_{i=1}^m a_i = l > 0$.

Theorem 1 tells us that there exist initial points in $(a_1, a_2, \cdots, a_m)$, if a_{r_1}, a_{r_s} $(s > 1)$ are two consecutive initial points, that is, a_{r_k} $(1 < k < s)$ is not initial point,we assert that $\sum_{i=1}^{s-1} a_{r_i} = 1$. Otherwise, $\sum_{i=1}^{s-1} a_{r_i} > 1$. Since $a_{r_1} = 1, \sum_{i=2}^{s-1} a_{r_i} \ge 1$. Now, let

$$S = \{k | \sum_{j=2}^k a_{r_j} = 0, \quad \text{and} \quad 2 \le k < s-1\},$$

$$h = \max S + 1, \text{ if } S \ne \emptyset; \ = 2, \text{ if } S = \emptyset.$$

Obviously, a_{r_h} is an initial point, contradicting the consecutivity of a_{r_1} and a_{r_s} (since $1 < h < s$). Hence, $\sum_{i=1}^{s-1} a_{r_i} = 1$. Since $\sum_{i=1}^m a_i = l$, there are exactly l initial points in $(a_1, a_2, \cdots, a_m)$.

Untying $(a_1, a_2, \cdots, a_m)$ at a_i, we obtain a line arrangement or sequence

$$\langle a_{i_1}, a_{i_2}, \cdots, a_{i_m} \rangle.$$

Let $p = \min\{q | a_{i_1} = a_{i_{1+q}}$ for all $i = 1, 2, \cdots .m\}$, then

$$(a_1, a_2, \cdots, a_m) = (a_1, \cdots, a_p, a_1, \cdots, a_p, \cdots, a_1. \cdots, a_p)$$

consists of $\frac{m}{p}$ sequences $\langle a_1, a_2, \cdots, a_p \rangle$, l initial points in $(a_1, a_2, \cdots, a_m)$ produce $\frac{l}{\frac{m}{p}} = \frac{lp}{m}$ different sequences.

Summing up the above discussion, we have the following

Theorem 2 *If $(a_1, a_2, \cdots, a_m)$ is any circle arrangement of integers with $a_i \leq 1$ for all i, and with $\sum_{i=1}^{m} a_i = l > 0$, then there are exactly l initial points, but exactly $\frac{lp}{m}$ of the cyclic shifts*

$$\langle a_1, a_2, \cdots, a_m \rangle, \langle a_2, \cdots, a_m, a_1 \rangle, \cdots, \langle a_m, a_1, \cdots, a_{m-1} \rangle$$

have all positive partial sums.

This is a correction of the generalization of Raney's lemma (see [1]) which says: If $\langle x_1, x_2, \cdots, x_m \rangle$ is any sequence of integers with $x_i \leq 1$ for all j, and with $x_1 + x_2 + \cdots + x_m = l \geq 0$, then exactly l of the cyclic shifts

$$\langle x_1, x_2, \cdots, x_m \rangle, \langle x_2, \cdots, x_m, x_1 \rangle, \cdots, \langle x_m, x_1, \cdots, x_{m-1} \rangle$$

have all positive partial sums.

For example, for given sequence $\langle -2, 1, 1, 1, -2, 1, 1, 1 \rangle, m = 8, l = 2, p = 4$, there is exactly $\frac{2 \times 4}{8} = 1$, but not two, cyclic shift $\langle 1, 1, 1, -2, 1, 1, 1, -2 \rangle$ which has all partial sums positive. Of course, there are 2 initial points in the circle arrangement $(-2, 1, 1, 1, -2, 1, 1, 1)$.

References:

[1] GRAHAM R L, KNUTH D F, PATASHNIK O. *Concrete Mathematics* [M]. Addison-Wesley Publishing Company, 1992, 345, 348.

关于 Raney 引理的修正与扩展

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摘 要: 本文对 Raney 引理进行了扩展，并对 R.L.Graham 等人的著作 Concrete Mathematics 中涉及的一个广义 Raney 引理进行了修正.