Journal of Mathematical Research & Exposition
Vol.21, No.4, 495-499, November, 2001

Construction of A Kind of Wavelets *

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Abstract: A method to construct wavelets on invariant sets is given, which can be used to construct nontensor-type multiwavelets.

Key words: wavelet; invariant set.

Classification: AMS(1991) 42A65/CLC O174.2

Document code: A **Article ID**: 1000-341X(2001)04-0495-05

In [2], the authors constructed wavelets on compact and invariant set in the sense of Hutchinson [1]. These wavelets were then used to give approximate solutions of integral equations defined on such sets. For the purpose of using the similar method to treat integral equations defined on unbounded sets. In this paper, we give the construction of wavelets on a kind of sets including those that are unbounded. Our method can be used to construct non-tensor-type multiwavelets.

In the following, for measurable set $A \subset B$, we take $f \in L^2(A)$ as function $\hat{f}: \hat{f}(x) = f(x), x \in A, \hat{f}(x) = 0, x \in B \setminus A$ and $L^2(A)$ as the subspace of $L^2(B)$.

Let $\varphi_i : R^d \to R^d, i = 0, 1, \ldots, \eta - 1$ be contractive affine maps for which the absolute value of their Jacobians are all equal to η^{-1} and assume that for measurable compact set $E \subset R^d, E = \bigcup_{i=0}^{\eta-1} \varphi_i(E)$, $\mathrm{m}(\varphi_i(E) \bigcap \varphi_j(E)) = \delta_{ij}\eta^{-1}\mathrm{m}(E)$ and $\varphi_i(E) = E_i$ are self-similar to $E, i, j = 0, 1, \ldots, \eta - 1$.

Now we choose maps $\psi_k, k = 0, 1, \ldots, \zeta - 1, \zeta \leq \eta$ which satisfy

(i) For any $k = 0, 1, \ldots, \zeta - 1$, there is a φ_i and a constant a_i such that $\psi_k = a_i \varphi_i$;

(ii) For any $i = 0, 1, \ldots, \eta - 1$, there is a ψ_k and a constant b_k such that $\varphi_i(E)$ is the translation of $b_k \psi_k(E)$;

(iii) $\mathrm{m}(\psi_i(E) \bigcap \psi_j(E)) = \delta_{ij}\mathrm{m}(E), i, j = 0, 1, \ldots, \zeta - 1$.

For $d = 2$, E can be chosen to be a triangle, rectangle, parallelgram or L-shaped region.

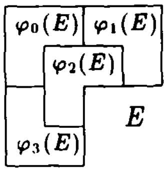

Fig.1

*Received date: 1999-03-05
Foundation item: Supported by the Science Foundation, the Returned Overseas Scholar Foundation and Academic Leading Persion Foundation of Shanxi Province.
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Fig.1 shows an L-shaped region for which $\psi_0 = 2\varphi_0 = I, \psi_1 = 2\varphi_1, \psi_2 = 2\varphi_3$.

Let $M_k \subset R^d, k = 0, 1, \ldots, \zeta - 1$, be index sets , $\psi_{k,u_1,\ldots,u_d}, (u_1, \ldots, u_d) \in M_k$, be translations of ψ_k. Choose M_k and the translations such that $\mathrm{m}(\psi_{i,u_1,\ldots,u_d}(E) \bigcap \psi_{j,v_1,\ldots,v_d}(E)) = \delta_{ij} \prod_{k=1}^{d} \delta_{u_k v_k} \mathrm{m}(E)$, $(u_1, \ldots, u_d) \in M_i, (v_1, \ldots, v_d) \in M_j, i, j \in \{0, 1, \ldots, \zeta - 1\}$. We will develop our theory on

$$D = \bigcup_{k=0}^{\zeta-1} \bigcup_{(u_1,\ldots,u_d) \in M_k} \psi_{k,u_1,\ldots,u_d}(E).$$

In the following, for simplicity suppore that $d = 2$. Then for $E = [0,1] \times [0,1]$, let $\eta = 4, \varphi_0(x,y) = (x/2, y/2), \varphi_1(x,y) = ((x+1)/2, y/2), \varphi_2(x,y) = (x/2, (y+1)/2), \varphi_3(x,y) = ((x+1)/2, (y+1)/2)$. In this situation, we have only one map $\psi_0 = 2\varphi_0 = I$. Thus $(-\infty, \infty) \times (-\infty, \infty) = \bigcup_{u=-\infty}^{\infty} \bigcup_{v=-\infty}^{\infty} \psi_{0,u,v}(E)$, $[0,m] \times [0,n] = \bigcup_{u=0}^{m-1} \bigcup_{v=0}^{n-1} \psi_{0,u,v}(E)$, where $\psi_{0,u,v}(x,y) = \psi_0(x+u, y+v)$.

Now choose an $\eta \times \eta$ orthogonal matrix $Q = (q_{ij}), i, j = 0, 1, \ldots, \eta - 1$ such that $Q^T Q = \eta I$ and define linear operators on $L^2(E)$

$$(T_i g)(t) = \sum_{j=0}^{\eta-1} q_{ij} g(\varphi_j^{-1}(t)) \chi_{E_j}(t), t \in E, i = 0, 1, \ldots, \eta - 1. \tag{1}$$

Let $\mathbf{f} = (f_1, \ldots, f_\xi)^T : E \to R^\xi$ be orthogonal refinable vector which satisfies a refinement equation $G_i \mathbf{f} = A_i^T \mathbf{f}$, where $G_i g = g \circ \varphi_i, g \in L^2(E)$, $G_i \mathbf{f}$ means that G_i acts on $\mathbf{f}$ componentwisely, and A_i is a $\xi \times \xi$ matrix, $\langle f_i, f_j \rangle = \delta_{ij}, i, j = 0, 1, \ldots, \eta - 1$. Moreover, there is a vector $\mathbf{v} \in R^\xi$ and $a \neq 0$ such that $\mathbf{v}^T \mathbf{f} = a$.

We say that $\mathbf{f}$ is an orthogonal refinable vector on E with respect to φ_i and $A_i, i = 0, 1, \ldots, \eta - 1$.

Suppose that

$$F_0 = \mathrm{span}_{L^2(D)}\{f_j \circ \psi_{k,u,v}^{-1} : j = 1, 2, \ldots, \xi, (u,v) \in M_k, k = 0, 1, \ldots, \zeta - 1\}, \tag{2}$$

$$F_n = \mathrm{span}_{L^2(D)}\{(T_{\varepsilon_n} \cdots T_{\varepsilon_1} f_j) \circ \psi_{k,u,v}^{-1} : \varepsilon_h \in \{0, 1, \ldots, \eta - 1\}, \\ h = 1, 2, \ldots, n, j = 1, 2, \ldots, \xi, (u,v) \in M_k, k = 0, 1, \ldots, \zeta - 1\}, n = 1, 2, \ldots. \tag{3}$$

Lemma 1 *Let*

$$F_{0,k,u,v} = \mathrm{span}_{L^2(\psi_{k,u,v}(E))}\{f_j \circ \psi_{k,u,v}^{-1} : j = 1, 2, \ldots, \xi\},$$

$$F_{n,k,u,v} = \mathrm{span}_{L^2(\psi_{k,u,v}(E))}\{(T_{\varepsilon_n} \cdots T_{\varepsilon_1} f_j) \circ \psi_{k,u,v}^{-1} : \varepsilon_h \in \{0, 1, \ldots, \eta - 1\}, \\ h = 1, 2, \ldots, n, j = 1, 2, \ldots, \xi\}, (u,v) \in M_k, k = 0, 1, \ldots, \zeta - 1, n = 1, 2, \ldots.$$

Then $F_n = \mathrm{span}_{L^2(D)} \{F_{n,k,u,v} : (u,v) \in M_k, k = 0, 1, \ldots, \zeta - 1\}, n = 0, 1, 2, \ldots$

Lemma 2 *Let ψ be an affine map on E, $\omega_i = \psi \circ \varphi_i \circ \psi^{-1}$, $i = 0, 1, \ldots, \eta - 1$. If $\mathbf{f}$ is an orthogonal refinable vector on E with respect to φ_i and $A_i, i = 0, 1, \ldots, \eta - 1$, then $\mathbf{f} \circ \psi^{-1}$*

is also an orthogonal refinable vector on $\psi(E)$ with respect to ω_i and $A_i, i = 0,1,\ldots,\eta-1$.

Proof $\langle f_i \circ \psi^{-1}, f_j \circ \psi^{-1}\rangle = \delta_{ij}$ can be verified directly. Let $G_i'g = g \circ \omega_i, g \in L^2(\psi(E))$. By definition of the orthogonal refinable vector one has $(G_i\mathbf{f})(x) = (\mathbf{f}\circ\varphi_i)(x) = (A_i^T\mathbf{f})(x)$. Let $x = \psi^{-1}(t)$, and notice that $\varphi_i \circ \psi^{-1} = \psi^{-1}\circ\omega_i$, then

$$(G_i\mathbf{f})(\psi^{-1}(t)) = (\mathbf{f}\circ\varphi_i)(\psi^{-1}(t)) = (\mathbf{f}\circ\psi^{-1})(\omega_i(t)) = (A_i^T\mathbf{f}\circ\psi^{-1})(t).$$

On the other hand one has $(G_i\mathbf{f})(\psi^{-1}(t)) = (\mathbf{f}\circ\psi^{-1})(\omega_i(t)) = (G_i'(\mathbf{f}\circ\psi^{-1}))(t)$. Therefore

$$G_i'(\mathbf{f}\circ\psi^{-1}) = (\mathbf{f}\circ\psi^{-1})\circ\omega_i = A_i^T\mathbf{f}\circ\psi^{-1}.$$

Let $\omega_{i,k,u,v} = \psi_{k,u,v}\circ\varphi_i\circ\psi_{k,u,v}^{-1}, i = 0,1,\ldots,\eta-1, (u,v)\in M_k, k = 0,1,\ldots,\zeta-1$, and define $T_{i,k,u,v}', T_i'$ on $L^2(D)$ as follows

$$(T_{i,k,u,v}'g)(t) = \sum_{j=0}^{\eta-1} q_{ij}\, g(\omega_{j,k,u,v}^{-1}(t))\chi_{\psi_{k,u,v}(E_j)}(t),$$
$$t\in D, i = 0,1,\ldots,\eta-1, (u,v)\in M_k, k = 0,1,\ldots,\zeta-1, \tag{4}$$
$$(T_i'g)(t) = \sum_{k=0}^{\zeta-1}\sum_{(u,v)\in M_k}(T_{i,k,u,v}'g)(t), t\in D, i = 0,1,\ldots,\eta-1. \tag{5}$$

Lemma 3 *$T_{i,k,u,v}'$ is a bounded linear operator on $L^2(D)$ and T_i' is an isometric linear operator on $L^2(D)$, $i = 0,1,\ldots,\eta-1, (u,v)\in M_k, k = 0,1,\ldots,\zeta-1$.*

Proof A direct check shows that $T_{i,k,u,v}'$ and T_i' are well defined for any $g\in L^2(D)$ and $t\in D$. By the fact that $Q^TQ = \eta I, |J_\psi| = 1, |J_{\varphi_i}| = \eta^{-1}$, where $|J_f|$ is the absolute value of Jacobian of f, $\|T_i'f\| = \|f\|$. □

Because on $\psi_{k,u,v}(E_j), (T_i'g)(t) = (T_{i,k,u,v}'g)(t)$, it cannot cause confusion if we note $\omega_{j,k,u,v}, T_{i,k,u,v}'$ by ω_j, T_i' , respectively.

Lemma 4 *Let*

$$F_{0,k,u,v}' = \mathrm{span}_{L^2(\psi_{k,u,v}(E))}\{f_j\circ\psi_{k,u,v}^{-1} : j = 1,2,\ldots,\xi\},$$
$$F_{n,k,u,v}' = \mathrm{span}_{L^2(\psi_{k,u,v}(E))}\{T_{\varepsilon_n}'\cdots T_{\varepsilon_1}'(f_j\circ\psi_{k,u,v}^{-1}) : \varepsilon_h\in\{0,1,\ldots,\eta-1\},$$
$$h = 1,2,\ldots,n, j = 1,2,\ldots,\xi\}, (u,v)\in M_k, k = 0,1,\ldots,\zeta-1, n = 1,2,\ldots$$

Then $F_{n,k,u,v}' = F_{n,k,u,v}, n = 0,1,\ldots$

Proof From the following and using recursive method we can get the result:

$$\begin{aligned}(T_{\varepsilon_2}T_{\varepsilon_1}f_i)\circ\psi_j^{-1} &= (T_{\varepsilon_2}(f_i\circ\varphi_{\varepsilon_1}^{-1}))\circ\psi_j^{-1} = (f_i\circ\varphi_{\varepsilon_1}^{-1}\circ\varphi_{\varepsilon_2}^{-1})\circ\psi_j^{-1}\\ &= f_i\circ(\varphi_{\varepsilon_1}^{-1}\circ\psi_j^{-1})\circ\omega_{\varepsilon_2}^{-1} = f_i\circ\psi_j^{-1}\circ\omega_{\varepsilon_1}^{-1}\circ\omega_{\varepsilon_2}^{-1}\\ &= (T_{\varepsilon_1}'(f_i\circ\psi_j^{-1}))\circ\omega_{\varepsilon_2}^{-1} = (T_{\varepsilon_2}'T_{\varepsilon_1}')(f_i\circ\psi_j^{-1}).\end{aligned}$$

Lemma 5 *Let* $\mathbf{g}=(g_1,\ldots,g_\xi)^T$ *be any orthogonal refinable vector on* E *with respect to contractive affine maps* θ_i *and matrices* $A_i, i=0,1,\ldots,\eta-1$. $F_0^0=\mathrm{span}_{L^2(E)}\{g_j: j=1,2,\ldots,\xi\}$, $F_n^0=\mathrm{span}_{L^2(E)}\{T_{\varepsilon_n}^0\ldots T_{\varepsilon_1}^0 g_j:\varepsilon_h\in\{0,1,\ldots,\eta-1\}, h=1,2,\ldots,n, j=1,2,\ldots,\xi\}$, $n=1,2,\ldots$ *Then*

$$F_{n+1}^0=\bigoplus_{i=0}^{\eta-1}T_i^0F_n^0,\ F_n^0\subset F_{n+1}^0, n=0,1,\ldots.$$

Let $W_n^0\bigoplus F_n^0=F_{n+1}^0$, *then*

$$W_{n+1}^0=\bigoplus_{i=0}^{\eta-1}T_i^0W_n^0,\quad L^2(E)=\overline{\bigcup_{n=0}^{\infty}F_n^0}=\overline{F_0^0\bigoplus\bigoplus_{n=0}^{\infty}W_n^0},$$

where T_i^0 *are the operators of (1) with* φ_j *being substitued by* θ_j *and* E_j *by* $\theta_j(E)$.

Lemma 5 can be deduced from [2 Th. 4.1, 4.2].

By Lemma 4, 5 we can develop wavelet structure directly from $F'_{n,k,u,v}$ using T'_i while we cannot do so from $F_{n,k,u,v}$ using T_i.

From Lemmas 1-5 we have

Theorem 1 $F_{n+1}=\bigoplus_{i=0}^{\eta-1}T'_iF_n$, $F_n\subset F_{n+1}, n=0,1,\ldots.$ *Let* $W_n\bigoplus F_n=F_{n+1}$, *then* $W_{n+1}=\bigoplus_{i=0}^{\eta-1}T'_iW_n$ *and*

$$L^2(D)=\overline{\bigcup_{n=0}^{\infty}F_n}=\overline{F_0\bigoplus\bigoplus_{n=0}^{\infty}W_n}. \tag{6}$$

Proof The main steps of the proof can be completed by Lemma 1-5. We only need to point out that if some of M_k are infinite sets, for which our integrals are over unbounded sets when we calculate the norms of the elemens in relative linear spans, our results can be obtained by the fact that an integral on an unbounded set can be approximated by one on bounded set. □

In the following, we generate the wavelet bases. Suppose that

$$\tilde{w}_1=-\sum_{j=1}^{\xi}\langle T_0f_1,f_j\rangle f_j+T_0f_1,$$

$$\tilde{w}_{\varepsilon\xi+\ell}=-\sum_{j=1}^{\xi}\langle T_0f_\ell,f_j\rangle f_j-\sum_{j=1}^{\varepsilon\xi+\ell-1}\langle T_\varepsilon f_\ell,w_j\rangle w_j+T_\varepsilon f_\ell,$$

where w_j is the unit function of $\tilde{w}_j$, $\varepsilon=0,1,\ldots,\eta-1,\ell=2,\ldots,\xi$. Let

$$\begin{aligned}\varphi_{0,j,k,u,v}&=f_j\circ\psi_{k,u,v}^{-1},\varphi_{1,j,k,u,v}=w_j\circ\psi_{k,u,v}^{-1}, j=1,2,\ldots\xi, k=0,1,\ldots,\zeta-1,\\ \varphi_{i,j,k,u,v}&=(T_{\varepsilon_{i-1}}\ldots T_{\varepsilon_2}T_{\varepsilon_1}w_\ell)\circ\psi_{k,u,v}^{-1}, j=(\eta^{i-2}\varepsilon_1+\ldots+\eta\varepsilon_{i-2}+\varepsilon_{i-1})\xi+\ell,\\ &\varepsilon_h\in\{0,1,\ldots,\eta-1\}, h=1,2,\ldots,i-1,\ell=1,2,\ldots,\xi,\\ &i=2,3,\ldots,k=0,1,\ldots,\zeta-1,u,v=\ldots,-1,0,1,\ldots.\end{aligned}$$

Theroem 2

$L^2(D) = \text{span}_{L^2(D)}\{\varphi_{i,j,k,u,v} : i = 0, 1, \ldots, j = 1, 2, \ldots, J(i), (u,v) \in M_k, k = 0, 1, \ldots, \zeta-1\}$,

where $J(0) = \xi, J(i) = \xi\eta^{i-1}, i \geq 1$.

Proof In short, we can prove that $\{\varphi'_{i,j,k,u,v}\}$ is a wavelet basis where $\varphi'_{i,j,k,u,v} = T'_{\epsilon_{i-1}} \ldots T'_{\epsilon_2} T'_{\epsilon_1}(w_{\ell} \circ \psi^{-1}_{k,u,v})$. But it is equal to $\varphi_{i,j,k,u,v}$ by Lemma 4. □

If $f \in L^2(D)$, then

$$f = \sum_{i=0}^{\infty} \sum_{j=1}^{J(i)} \sum_{k=0}^{\zeta-1} \sum_{(u,v)\in M_k} c_{i,j,k,u,v}\varphi_{i,j,k,u,v},$$

where $c_{i,j,k,u,v} = \langle f, \varphi_{i,j,k,u,v}\rangle, i = 0, 1, \ldots, j = 1, 2, \ldots, J(i), (u,v) \in M_k, k = 0, 1, \ldots, \zeta - 1$. The series converges in the norm of $L^2(D)$.

For $(-\infty, \infty) \times (-\infty, \infty)$, we can choose the set of orthogonal polynomials $\{f_1, \ldots, f_\xi\}$ on $[0,1] \times [0,1]$ to form **f** and give the relative wavelet basis.

It is evident that our wavelets can be non-tensor-type.

The simplist example of our wavelet is the Harr wavelet.

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一类小波的构造

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摘 要: 给出基于自相似形定义于无界集的多元小波构造方法，利用此方法，可以构造非张量积形式的多元小波.