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Remarks on Asymptotic Expansion of C_0-Semigroups *

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Abstract: In this paper, we consider the conditions of asymptotic expansion for C_0-semigroups and obtain a general result. Finally, we give an application to neutron transport equation.

Key words: C_0-semigroup; resolvent; asymptotic; expansion.

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1. Introduction

Let X be a Banach space, A is a generator of C_0-semigroup $T(t)$. $\rho(A)$ denotes the resolvent set of $A, \sigma(A)$ the spectral set of A and $R(\lambda, A)$ the resolvent of A.

In the paper [1], the authors obtain the following theorem for the asymptotic expansion of C_0-semigroup.

Theorem (*) *Let $T(t)$ be a C_0-semigroup in Banach space X with the generator A. There exists a constant β, such that the right half plane of the line $\mathrm{Re}\lambda = \beta_0$ in the complex plane C is only composed of at most denumerably many points which are eigenvalues with finite multiplicity. Furthermore, $\lambda_1, \lambda_2, \cdots$ with $\mathrm{Re}\lambda_n \geq \lambda_{n+1}, n = 1, 2, 3, \cdots$, for $x \in D(A^2)$ and $\sigma > \beta_0$,*

$$\lim_{|\tau|\to\infty} \|R(\sigma + i\tau, A)x\| = 0$$

uniformly for σ.

Then, $\forall x \in D(A^2)$

$$T(t)x = \sum_{n=1}^{m} T_n(t)x + R_m(t)x,$$

where $T_n(t)x = \mathrm{Res}(\exp(\lambda t)R(\lambda, A)x)|_{\lambda=\lambda_n}$ denotes the residue value of $\exp(\lambda t)R(\lambda, A)x$ in $\lambda = \lambda_n$. $\|R_m(t)\| \leq P\exp((\mathrm{Re}\lambda_m - \varepsilon)t), t \geq 0$. Here $\varepsilon > 0, P > 0$ are constants and $\mathrm{Re}\lambda_m - \varepsilon > \mathrm{Re}\lambda_{m+1}$.

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In the present paper, we also discuss the problem on the asymptotic expansion of C_0-semigroup, we obtain a general result.

2. Asymptotic expansion of C_0-semigroup

Lemma 2.1[2] *Let $\omega > 0, F(\mu) : (\omega, \infty) \to X$. Suppose that $F(\mu)$ have the following the representation of Laplace-Stieljes: $F(\mu) = \mu \int_0^\infty e^{-\mu t}\alpha(t)\mathrm{d}t$. $\alpha(0) = 0$ and $\|\alpha(t+h) - \alpha(t)\| \le Mhe^{\omega(t+h)}$ for $t, h \ge 0$, then*

$$\alpha(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\mu t}\frac{F(\mu)}{\mu}\mathrm{d}\mu, \gamma > \omega \tag{2.1}$$

converges uniformly in $t \in [R^{-1}, R], R > 0$.

Theorem 2.2 *Let operator A generate a C_0-semigroup $T(t)$ satisfing $\|T(t)\| \le Me^{\omega t}$. Then $\forall x \in X$*

$$\int_0^t T(s)x\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{(\lambda - A)^{-1}x}{\lambda}\mathrm{d}\lambda, \gamma > \omega, \tag{2.2}$$

and $\forall x \in D(A)$

$$T(t)x = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - A)^{-1}x\mathrm{d}\lambda, \gamma > \omega. \tag{2.3}$$

Proof Since $\|T(t)\| \le Me^{\omega t}$, set $\alpha(t) = \int_0^t T(s)x\mathrm{d}s, \forall x \in X$, it is easy to see that $\alpha(t)$ satisfies the condition in Lemma 1.

Noting

$$(\lambda - A)^{-1}x = \lambda\int_0^\infty e^{-\lambda t}\left(\int_0^t T(s)x\mathrm{d}s\right)\mathrm{d}t, \mathrm{Re}\lambda > \omega,$$

we get (2.2).

On the other hand, for $x \in D(A)$, by (2.2)

$$\int_0^t T(s)Ax\mathrm{d}s = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}\frac{(\lambda - A)^{-1}Ax}{\lambda}\mathrm{d}\lambda, \gamma > \omega,$$

we have

$$\begin{aligned}\int_0^t T(s)Ax\mathrm{d}s &= -\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}x\mathrm{d}\lambda + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - A)^{-1}x\mathrm{d}\lambda \\ &= -x + \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - A)^{-1}x\mathrm{d}\lambda.\end{aligned}$$

Then

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda - A)^{-1}x\mathrm{d}\lambda = x + \int_0^t T(s)Ax\mathrm{d}s,$$

noting $\forall x \in D(A)$

$$T(t)x = x + \int_0^t T(s)Ax\mathrm{d}s.$$

The proof is complete.

Theorem 2.3 *Let $T(t)$ be a C_0-semigroup in Banach space X, with generator A. The right half plane of the line* $\mathrm{Re}\lambda = \beta_0$ *is only composed of at most denumerably many points which are eigenvalues with finite multiplicity of $A, \lambda_1, \lambda_2, \cdots. \forall x \in X$ and $\sigma > \beta_0$, uniformly for σ, we have*

$$\lim_{|\tau| \to \infty} \|R(\sigma + i\tau, A)x\| / |\tau| = 0,$$

then $\forall x \in D(A)$

$$T(t)x = \sum_{n=1}^{m} T_n(t)x + R_m(t)x.$$

Proof Using Theorem 2, $\forall x \in D(A)$, we have

$$T(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t} (\lambda - A)^{-1} x \mathrm{d}\lambda.$$

On the other hand, since $\lambda R(\lambda, A)x = x + R(\lambda, A)Ax$, then $\|R(\sigma + i\tau, A)x\| \le \frac{\|x\|}{|\tau|} + \frac{\|R(\sigma + i\tau, A)Ax\|}{|\tau|}$.

It is easy to prove $\forall x \in D(A), \sigma \ge \beta_0$

$$\lim_{|\tau| \to \infty} \|R(\sigma + i\tau, A)x\| = 0.$$

The following process is similar to [1], we omit it.

Similar to Theorem 2.3, we have the following result.

Theorem 2.4 *Let A be a generator of C_0-semigroup $T(t)$ satisfing the conditions in Theorem 2.3, furthermore, $\forall x \in X$ and $\sigma > \beta_0$,*

$$\lim_{|\tau| \to \infty} \|R(\sigma + i\tau, A)x\| / \tau^2 = 0$$

umiformly for σ. Then $\forall x \in D(A^2)$

$$T(t)x = \sum_{n=1}^{m} T_n(t)x + R_m(t)x.$$

3. Example

We consider the following neutron transport equation ([3],[4]):

$$\begin{cases} \dfrac{\partial f(x,\mu,t)}{\partial t} = -\mu \dfrac{\partial f(x,\mu,t)}{\partial x} - \sigma(x) f(x,\mu,t) + \displaystyle\int_{-1}^{1} k(x,\mu,\mu') f(x,\mu',t) \mathrm{d}\mu', \\ f(-a,\mu,t) = 0, \ 0 \le \mu \le 1, \\ f(a,\mu,t) = 0, \ -1 \le \mu \le 0, \\ f(x,\mu,0) = f_0(x,\mu). \end{cases} \tag{3.1}$$

Set $X = [-a, a] \times [-1, 1]$, and let $L^2(X)$ be the Banach space of all square-integrable complex functions defined on X, define operators on $L^2(X)$ as follows:

$$Bf(x,\mu) = -\mu\frac{\partial f(x,\mu)}{\partial x} - \sigma(x)f(x,\mu),$$
$$Kf(x,\mu) = \int_{-1}^{1} k(x,\mu,\mu')f(x,\mu')\mathrm{d}\mu',$$
$$A = B + K,$$

where $D(B) = \{f : Bf \in L^2(X), f(-a,\mu) = 0, 0 \le \mu \le 1; f(a,\mu) = 0, -1 \le \mu \le 0\}$, $D(K) = L^2(X)$, thus $D(A) = D(B)$.

Then, equation (3.1) can be written as

$$\frac{\mathrm{d}f(t)}{\mathrm{d}t} = Af(t), \quad f(0) = f_0. \tag{3.2}$$

By the results of [3], [4], for A and B we have

Theorem 3.1 $\lambda^* = \inf_{-a\le x\le a}\sigma(x)$, *then*

(1) $\{\lambda : \mathrm{Re}\lambda > -\lambda^*\} \subset \rho(B)$*;*

(2) $\{\lambda : \mathrm{Re}\lambda > \|K\| - \lambda^*\} \subset \rho(A)$*;*

(3) Let b_1, b_2 *be two real numbers satisfing* $-\lambda^* < b_1 < \|K\| - \lambda^* < b_2$, *then,* A *has at most finite eigenvalues with finite algebraic multiplicity in the strip* $\{\lambda : b_1 \le \mathrm{Re}\lambda \le b_2\}$*;*

(4) A *generates a* C_0*-semigroup* $T(t)$ *with* $\|T(t)\| \le Me^{\omega t}$.

By the consequence in section 2 of the paper, we obtain immedily the following theorem:

Theorem 3.2 *For* $f_0(x,\mu) \in D(A)$, *the neutron transport equation (3.1) has unique solution:*

$$f(x,\mu,t) = T(t)f_0(x,\mu).$$

For constant b_1 *with* $-\lambda^* < b_1 < \|K\| - \lambda^*$, *the solution of equation (3.1) can be expanded as follows:*

$$f(x,\mu,t) = \sum_{j=1}^{m} T_m(t)f_0(x,\mu) + o(e^{b_1 t}),$$

where $T_m(t)f_0(x,\mu) = \mathrm{Res}(\exp(\lambda t)R(\lambda, A)f_0(x,\mu))|_{\lambda=\lambda_m}\lambda_1, \lambda_2, \cdots, \lambda_m$ *are the eigenvalues of* A *with* $\mathrm{Re}\lambda_k \ge \lambda_{k+1}, k = 1, 2, \cdots, m-1$ *and* $\mathrm{Re}\lambda_k \ge b_1, k = 1, 2, \cdots, m$.

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关于 C_0 半群渐近展开的几点注记

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摘 要: 本文考察了 C_0 半群渐近展开的一些成立条件，得到了一个较一般的结果. 最后，给出了它在中子迁移方程中的应用.