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A Note on Functor ( )^0 *

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Abstract: We construct a counter-example about functor ( )^0, and prove an isomorphism theorem from convolution algebra to dual algebra of tensor coalgebra.

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1. A counter-example

All discussions are over a fixed field k, and we assume the basic Hopf algebras, see [1].

For arbitary coalgebra C, C^* is always an algebra. But A^* is not a coalgebra for an algebra A in general. The reason is: if $M: A\otimes A \to A$ is product, then $M^*: A^* \to (A\otimes A)^*$. When A is finite dimensional, $(A\otimes A)^* \cong A^* \otimes A^*$ and M^* is coproduct over A^*. But for infinite dimensional $A, A^* \otimes A^*$ is isomorphic to proper subspace of $(A \otimes A)^*$. In general, $M^*(A^*) \subseteq A^* \otimes A^*$ is not satisfied. Therefore it is not sure whether A^* is a coalgebra. Anyway, A^* contains a "maximum subcoalgebra" A^0, which is known as dual coalgebra of A, and the following statements are equivalent:

(1) $A^0 = M^{*-1}(A^* \otimes A^*)$.

(2) $A^0 = \{a^* \in A^* | \ker a^* \text{ contains confinite ideals of } A\}$.

A cofinite dimensional ideal I is nothing but A/I finite dimension.

Functor ( )^0 ([1], chapter VI) is applied to wide branches such as reflexivity (see [1-3]), quantum groups (see [4]), etc. Function ( )^0 has the following basic property ([1], chapter VI):

Proposition *Let A, B be algebras. If $f: A \to B$ is algebra morphism, then $f^*: B^* \to A^*$ satisfies $f^*(B^0) \subseteq A^0$.*

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It is natural to ask whether conclusion $f^*(B^0) \subseteq A^0$ is true if $f \in \mathrm{Hom}_k(A, B)$ is merely k-linear. The answer is negative. The counter-example is technically constructed as follows:

Let $A = k[x, x^{-1}]$ be a Laurent polynomial algebra, $B = k[x]$ a polynomial algebra.

(1) Any non-zero ideal I of A contains at least one polynomial with non-zero constant term.

In fact, let $\varphi(x) = \sum_{i=-m}^{n} a_i x^i \in I, \varphi(x) \neq 0$. Then $\varphi(x)$ has at least one cofficient $a_k \neq 0$. If $k = 0$, then $\varphi(x)$ is desired. Otherwise, $x^{-k}\varphi(a)$ is qualified.

(2) Construct a morphism $f : A \to B$ by "eliminating negative monomials", i.e,

$$f : \sum_{i=-m}^{n} a_i x^i \mapsto \sum_{i=0}^{n} a_i x^i, \quad \forall \sum_{i=-m}^{n} a_i x^i \in A,$$

Obviously, f is k-linear, not algebraic.

(3) Let $b^* \in B^*, b^* : B \to k$ satisfy $b^*(\varphi(x)) = \varphi(0), \quad \forall \varphi(x) \in B$.

Then f is k-linear surjective morphism from B to k, and that $\ker b^* = (x)$ is a principal idea generated by x. Furthermore, $B/\ker b^* \cong k$ by basic isomorphism theorems, and this means $\ker(b^*)$ has one codimension. Surely, $b^* \in B^0$.

(4) $f^*(b^*) \in A^*$ but $f^*(b^*) \notin A^0$. In fact, $\forall \sum_{i=-m}^{n} a_i x^i \in A^*, f^*(b^*) : \sum_{i=-m}^{n} a_i x^i \mapsto a_0$. Therefore, $\ker f^*(b^*) = \{\varphi(x) |$ constant term is zero$\} \neq 0$, which contains no any ideal according to (1). Of course, there is no any cofinite in $\ker f^*(b^*)$, i.e $f^*(b^*) \notin A^0$.

2. On reflexive algebras

For any algebra A, A^{0^*} is also an algebra. If natural morphism $\xi_A : A \to A^{0^*}$ is an algebra isomorphism, then A is called reflexive ([3]). As an application, we set up an isomorphism theorem from convolution algebra to dual algebra of tensor coalgebras.

Theorem *Let C be a coalgebra, A a reflexive algebra. Then convolution algebra $\mathrm{Hom}(C, A)$ is isomorphic to $(C \otimes A^0)^*$, the dual algebra of tensor coalgebras.*

Proof The proof will be given in several steps.

(1) Let $\varphi : \mathrm{Hom}(C, A) \to \mathrm{Hom}(C, A^{0^*})$ defined by $\varphi(g) = \xi_A g, \quad \forall g \in \mathrm{Hom}(C, A)$.

Obviously, φ is k-linear bijection. Moreover, φ is algebra isomorphism. In fact, $\forall g, h \in \mathrm{Hom}(C, A)$

$$\begin{aligned} g * h &= M(g \otimes h)\Delta, \\ \varphi(g * h) &= \xi_A(g * h) = \xi_A M(g \otimes h)\Delta \\ &= \overline{M}(\xi_A \otimes \xi_A)(g \otimes h)\Delta \quad (\xi_A \text{ is algebraic: } \xi_A M = \overline{M}(\xi_A \otimes \xi_A)) \\ &= \overline{M}(\xi_A g \otimes \xi_A h)\Delta = \varphi(g) * \varphi(h), \end{aligned}$$

where M and Δ a product and coproduct respectively, etc.

(2) Let $\Psi : (C \otimes A^0)^* \to \mathrm{Hom}(C, A^{0^*})$ be as follows:

$$\langle \Psi(f)(c), a^0 \rangle = \langle f, c \otimes a^0 \rangle, \quad \forall f \in (C \otimes A^0)^*, c \in C, a^0 \in A^0,$$

where bractet $\langle , \rangle$ means evaluation. Similar to proposition 3.8 of [6], one can prove Ψ is k-linear bijection. Furthermore, Ψ is algebra isomorphism. Note that

$$\langle \Psi(fg)(c), a^0 \rangle = \langle fg, c \otimes a^0 \rangle = \sum_{(c),(a^0)} \langle f, c_{(1)} \otimes a^0_{(1)} \rangle \langle g, c_{(2)} \otimes a^0_{(2)} \rangle,$$

where $f, g \in (C \otimes A^0)^*, c \in C, a^0 \in A^0$.

On the other hand,

$$\begin{aligned}\langle \Psi(f) * \Psi(g)(c), a^0 \rangle &= \sum_{(c)} \langle \Psi(f)(c_{(1)}) \Psi(g)(c_{(2)}), a^0 \rangle \\ &= \sum_{(c),(a^0)} \langle \Psi(f)(c_{(1)}), a^0_{(1)} \rangle \langle \Psi(g)(c_{(2)}), a^0_{(2)} \rangle \\ &= \sum_{(c),(a^0)} \langle f, c_{(1)} \otimes a^0_{(1)} \rangle \langle g, c_{(2)} \otimes a^0_{(2)} \rangle,\end{aligned}$$

$\Psi(fg) = \Psi(f) * \Psi(g)$ holds.

(3) It follows that $\Psi^{-1}\varphi$ is isomorphism of $\mathrm{Hom}(C, A)$ to $(C \otimes A^0)^*$ from (1), (2) above.

This completes the proof.

We show an example of the theorem as follows:

Example 1 If A is a finite dimensional algebra, and C is any coalgebra, then $\mathrm{Hom}(C, A) \cong C^* \otimes A$ as algebras.

Proof Certainly that finite dimensional A is reflexive and $A^0 \cong A^*, A \cong A^{**} \cong A^{0^*}$ (as algebra). $\mathrm{Hom}(C, A) \cong (C \otimes A^0)^* \cong C^* \otimes A^{0^*} \cong C^* \otimes A$ A^0, since is finite dimensional.

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关于函子$(\ )^0$的注记

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摘 要: 给出了一个对偶余代数问题的反例，作为反射代数的一个应用，建立从卷积代数 $\mathrm{Hom}(C, A)$ 到余代数张量的对偶代数 $(C \otimes A^0)^*$ 的同构映射.