Journal of Mathematical Research & Exposition
Vol.22, No.1, 7–12, February, 2002

On the Dimension for Multivariate Weak Spline Function Spaces *

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Abstract: In this paper, the dimension formulaes of multivariate weak spline are discussed. The dimension formulaes of non-degree multivariate weak spline on a vertex are presented. The dimension formulaes on triangulation are also discussed. At last, the local supported bases of $W_3^1(I_1\Delta)$ are presented.

Key words: spline; weak spline; dimension.

Classification: AMS(2000) 41A15,65D07/CLC O174.51

Document code: A **Article ID:** 1000-341X(2002)01-0007-06

1. Introduction

It is well known that multivariate spline is a kind of fundamental tool for computational geometry, numerical analysis, approximation, and optimization. But one have often met a new kind of spline that is named as multivariate weak spline. The multivariate weak spline is defined by a piecewise polynomial which smooths only on a set of discrete points[7]. The multivariate weak spline is important in finite element and CAGD and it is defined and discussed in [7,8]. In [7], the smooth condition and conformality condition about multivariate weak spline are presented. In [8], the B-net method for studying multivariate weak spline is discussed. In this paper, we will present some dimension formulations about multivariate weak spline spaces. To help make this paper self-contained the remainder of this section contains a brief statement of the relevant definitions. In this paper, $\mathbf{P_k(x,y)}$ denotes the collection of all bivariate polynomials of real coefficients with total degree k, and $\mathbf{P_k(x)}$ denotes the collection of all univariate polynomials of real coefficients with total degree k. $D^mP(x_i,y_i)$ denotes the collection of $\frac{\partial^m p(x,y)}{\partial^t x\partial^{m-t}y}|_{(x_i,y_i)}, 0\le t\le m$.

Definition 1 *Let l be a line segment, and the point set $S=\{(x_1,y_1),(x_2,y_2),\cdots,(x_n,y_n)\}$ lie on l. If $|S|:=card(S)$ is limited, and each point of S is an interior point of*

*Received date: 2000-02-25
Foundation item: Project supported by the National Natural Science Foundation of China (19871010, 69973010)
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the line segment l, then S is called an appointed point set,and every point in S is called an appointed point.

Let D be a domain in R^2, Δ be a partition of the domain D consisting of finite straight lines or line segments. An appointed point set is given on each inner edge. This partition is called an appointed point partition which is denoted by $I\Delta$. Let D_i, $i = 1, \cdots, T$, be all of the cells of Δ, and $S_j, j = 1, \cdots, L$ be all of the appointed point sets of $I\Delta$. Denote by $C^{\mu}_{(0)}(S)$ the set of functions with μ smoothness in each point of the given point set S. For integer $k > \mu \geq 0$,

$$W^{\mu}_k(I\Delta) = \{W(x,y) \in C^{\mu}_{(0)}(S) | W(x,y)|_{D_i} \in \mathbf{P_k(x,y)}, \forall D_i, S = \cup_j S_j\}$$

is called a multivariate weak spline space with degree k and smoothness μ, $W(x,y)|_D$ denotes the restriction of $W(x,y)$ on D. Let $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_L)$ is a L dimension vector. Then

$$W^{\vec{\mu}}_k(I\Delta) = \{W(x,y) \in \cap_{m=1}^{L} C^{\mu_m}_{(0)}(S_m) : W(x,y)|_{D_i} \in \mathbf{P_k(x,y)}, \forall D_i\}$$

is called a non-degree multivariate weak spline space with degree k and smoothness $\vec{\mu}$.

Theorem 1[5] *Denote by l the straight line: $y - ax - b = 0$. Let $p_1(x,y), p_2(x,y) \in \mathbf{P_k(x,y)}$, and S be a set of points on the line l.Let $p(x,y) = p_1(x,y) - p_2(x,y)$. Then*

$$D^j p(x_i, y_i) = 0, j = 0, \cdots, \mu, \forall (x_i, y_i) \in S$$

if and only if there exist $q(x,y) \in \mathbf{P_{k-\mu-1}(x,y)}$, and $c_m(x) \in \mathbf{P_{k-\mu-(|S|-1)m-1}(x)}$ such that

$$p(x,y) = (y - ax - b)^{\mu+1} q(x,y) + \sum_{m=1}^{\mu+1} (y - ax - b)^{\mu+1-m} \left(\prod_{i=1}^{|S|} (x - x_i)\right)^m c_m(x) \quad (1)$$

where $c_m(x) \equiv 0$ provided $k - \mu - (|S| - 1)m - 1 < 0$.

$q(x,y)$ is called to be the smoothing cofactor on the line l, and $c_m(x)$ is called to be the weak smoothing cofactor of order m on the line l. Suppose that N edges passing through the interior vertex v. Let S_i be the set of appointed points on the ith edge: $y - a_i x - b_i = 0$. Similarly as the ordinary multivariate spline,the following conformality condition at the interior vertex v

$$H_v = \sum_{i=1}^{N} ((y - a_i x - b_i)^{\mu+1} q_i(x,y) + \sum_{m=1}^{\mu+1} (y - a_i x - b_i)^{(\mu+1-m)} (\prod_{(x_i,y_i) \in S_i} (x - x_i))^m c_{mi}(x)) = 0$$

holds.

The following conditions are called to be global conformality conditions

$$H_{v0} = 0, H_{v1} = 0, \cdots, H_{vn} = 0 \quad (2)$$

where $\{v0, v1, \cdots, vn\}$ is the set of interior vertices of Δ. We have

Theorem 2[7] *Let $I\Delta$ be the appointed partition on D. $S(x,y) \in W_k^{\mu}(I\Delta)$ if and only if there exist the smoothing cofactor, the weak smoothing cofactor with order $m (1 \le m \le \mu+1)$, and the global conformality condition are satisfied.*

In this paper, we only consider a simple but practical case, i.e. the cardinality of appointed point set on each grid segment is 1 and Δ is a triangulation of a simply connected domain $D \subset R^2$. Denote the appointed point triangulation as $I_1\Delta$. The multivariate weak spline and non-degree multivariate weak spline over $I_1\Delta$ are denoted by $W_k^{\mu}(I_1\Delta)$ and $W_k^{\vec{\mu}}(I_1\Delta)$ respectively.

2. The dimension formulation for non-degree multivariate weak spline living on a vertex

Denote by $St(v)$ the collection of cells in Δ sharing v as a common vertex. Let N be the number of lines and (x_i, y_i) be appointed point lying on the ith grid line. The cell between the ith grid segment and the $(i+1)$th grid segment is called the ith cell, where $1 \le i \le N-1$; The cell between the Nth grid segment and 1th grid segment is called the Nth cell. Denote by $W_k^{\vec{\mu}}(St(v))$ the non-degree multivariate weak spline space over $St(v)$, where $\vec{\mu} = (\mu_1, \mu_2, \cdots, \mu_N)$. In this paper, we will consider the $\dim W_k^{\vec{\mu}}(St(v))$. Let $\mu_{\min} = \min\{\mu_i, i = 1, 2, \cdots, N\}, \mu_{\max} = \max\{\mu_i, i = 1, 2, \cdots, N\}$. We have

Theorem 3 *If $k \ge \mu_{min} + \mu_{max} + 1$, then $dim W_k^{\vec{\mu}}(St(v)) = NC_{k+2}^2 - \sum_{i=1}^{N} C_{\mu_i+2}^2$ where N is the number of grid lines.*

Proof Select an appropriate reference frame such that $(x_1, y_1) = (0,0), x_N \ne 0$. Without loss of generality suppose $\mu_N = \mu_{min}$. Suppose that the polynomial defined on the ith cell is $P_i(x,y) = \sum_{n+m \le k} a_{nm}^{(i)} x^n y^m$. Let $u_{nm}^{(i)} = \frac{\partial^{n+m} W(x_i, y_i)}{\partial x^n \partial y^m}, n+m \le \mu_i$, where $W(x,y)$ is the multivariate weak spline concerning the vertex v. According to the definition of $W_k^{\mu}(St(v))$, u_{nm}^{i} and $a_{nm}^{(i)}$ are unknown variables, we get the coefficient matrix

$$A = \begin{pmatrix} A_1 & 0 & 0 & 0 & -E & 0 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & A_3 & 0 & 0 & 0 & 0 & -E & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & \ddots & 0 & -E \\ 0 & 0 & 0 & A_N & 0 & 0 & 0 & 0 & -E \\ 0 & 0 & 0 & A_1 & -E & 0 & 0 & 0 & 0 \end{pmatrix},$$

where E is the identity matrix of order $C_{\mu_i+2}^2$, each A_i is a matrix of order $C_{\mu_i+2}^2 \times C_{k+2}^2$. The row vector of A_i is $\frac{\partial^{n+m} \mathbf{b}(\mathbf{x_i}, \mathbf{y_i})}{\partial x^n \partial y^m}, n+m \le \mu_i$, where $\mathbf{b}(\mathbf{x}, \mathbf{y}) = (x^k, x^{k-1}y, \cdots, y^k, x^{k-1}, \cdots, x, y, 1)$. It is clear that A_i is a row full rank matrix. After a series of transformations, the matrix A becomes:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -E \\ 0 & A_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & A_N & 0 & 0 & 0 & 0 \\ 0 & A_1 & A_1 & A_1 & A_1 & 0 & 0 & 0 & \end{pmatrix}.$$

Obviously, if $B = \binom{A_N}{A_1}$ is row full rank, then the matrix is also row full rank. Now to prove B is row full rank for $k \geq \mu_{min} + \mu_{max} + 1$. A_i can be written by $(\overline{A_i}, \overline{\overline{A_i}})$,where $\overline{\overline{A_i}}$ is a $C^2_{\mu_i+2} \times C^2_{\mu_{max}+2}$ submatrix. It is easy to prove that $\overline{\overline{A_i}}$ is a nonsingular matrix. Because of $(x_1, y_1) = (0, 0)$, so $\overline{A_1} = 0$. Since

$$B = \begin{pmatrix} A_N \\ A_1 \end{pmatrix} = \begin{pmatrix} \overline{A_N} & \overline{\overline{A_N}} \\ \overline{A_1} & \overline{\overline{A_1}} \end{pmatrix},$$

we need only prove $\overline{A_N}$ is row full rank. Obviously, if $\overline{A_N}$ is row full rank for $k = \mu_{max} + \mu_{min} + 1$ then $\overline{A_N}$ is also row full rank for $k > \mu_{max} + \mu_{min} + 1$. Now we prove $\overline{A_N}$ to be row full matrix for $k = \mu_{max} + \mu_{min} + 1$. After a series of primary transformations, $\overline{A_N}$ becomes

$$\begin{pmatrix} B_0 & D & E & F & C_0 \\ 0 & B_1 & G & H & C_1 \\ 0 & 0 & \ddots & K & \vdots \\ 0 & 0 & 0 & B_{\mu_{min}+1} & C_{\mu_{min}+1} \end{pmatrix},$$

where

$$B_0 = \begin{pmatrix} x_N^k & x_N^{k-1} & \cdots & x_N^{\mu_{max}+1} \\ C_k^1 x_N^{k-1} & C_{k-1}^1 x_N^{k-2} & \cdots & C^1_{\mu_{max}+1} x_N^{\mu_{max}} \\ \vdots & \vdots & \vdots & \vdots \\ \mu_{min}! C_k^{\mu_{min}} x_N^{k-\mu_{min}} & \mu_{min}! C_{k-1}^{\mu_{min}} x_N^{k-\mu_{min}-1} & \cdots & \mu_{min}! C^{\mu_{min}}_{\mu_{max}+1} x_N^{\mu_{max}-\mu_{min}+1} \end{pmatrix}.$$

Since $x_N \neq 0$, it is clear that B_0 is a row full rank matrix. Similarly $B_1, B_2, \cdots, B_{\mu_{min}+1}$ are row full rank matrices. Hence $\overline{A_N}$ is a row full rank matrix. Therefore the matrix A is row full rank matrix. The number of unknown variables is $NC^2_{k+2} + \sum_{i=1}^N C^2_{\mu_i+2}$ and the rank of coefficient matrix is $2\sum_{i=1}^N C^2_{\mu_i+2}$. So the dimension of the space $W_k^{\vec{\mu}}(St(v))$ is $NC^2_{k+2} - \sum_{i=1}^N C^2_{\mu_i+2}$. □

3. The dimension formulation for multivariate weak spline on triangulation

In the section, we will presented the dimension of $W_d^{\mu}(I_1\Delta), d \geq 2\mu+1$. We firstly introduce some notations. For triangulation Δ, Let T=number of triangles, E_I=number of interior edges, E_B=number of boundary edges, L= number of grid segments.

Firstly, we introduce the dimension of $W_{2\mu+1}^{\mu}(I_1\Delta)$. Consider the following problem.

$$\begin{aligned}
D^{m,n}P(x_1,y_1) &= D^{m,n}f(x_1,y_1)\\
D^{m,n}P(x_2,y_2) &= D^{m,n}f(x_2,y_2)\\
D^{m,n}P(x_3,y_3) &= D^{m,n}f(x_3,y_3), \quad m+n \leq \mu\\
D^{m,n}P(x_0,y_0) &= D^{m,n}f(x_0,y_0), \quad m+n \leq \mu-1
\end{aligned} \tag{3}$$

where $(x_1,y_1),(x_2,y_2),(x_3,y_3)$ are three vertices of ΔABC, (x_0,y_0) is an inner point of the triangle, $D^{m,n}f(x_i,y_i)$ denotes $\frac{\partial^{m+n}}{\partial x^m \partial y^n}f(x_i,y_i)$, $f(x,y) \in C^{\mu}$, $P(x,y) \in \mathbf{P}_{2\mu+1}(\mathbf{x},\mathbf{y})$.

We have

Lemma 1[7] *The system (3) has a unique solution.*

Theorem 4[7] $\dim(W_{2\mu+1}^{\mu}(I_1\Delta)) = TC_{2\mu+3}^2 - E_I C_{\mu+2}^2$.

Now we consider the dimension of $W_d^{\mu}(I_1\Delta), d \geq 2\mu+1$.

Theorem 5 *For* $d \geq 2\mu+1$, $\dim W_d^{\mu}(I_1\Delta) = TC_{d+2}^2 - E_I C_{\mu+2}^2$.

Proof According to the definition of multivariate weak spline,we know $w(x,y)|_{D_i} \in \mathbf{P}_d(x,y)$, where $w(x,y) \in W_d^{\mu}(I_1\Delta)$. Let $p_i(x,y) = w(x,y)|_{D_i}$ and $p_i(x,y) = \sum_{m+n\leq d} a_{mn}^{(i)} x^m y^n$. Suppose $D_{i_l}, 1 \leq l \leq 3$ share an interior edge with D_i. Denote the appointed point shared by D_i and D_{i_m} as $(x_{i_m}, y_{i_m}), 1 \leq m \leq l$. We have

$$D^h p_i(x_{i_m}, y_{i_m}) = D^h p_{i_m}(x_{i_m}, y_{i_m}), h = 0,\cdots,\mu, i = 1,\cdots,T. \tag{4}$$

Denote the solution space of system (4) as H. Obviously, $\dim W_d^{\mu}(I_1\Delta) = \dim H$. There are TC_{d+2}^2 unknown variables and $E_I C_{\mu+2}^2$ linear equations in system (4). The coefficient matrix of the linear equations is denoted as H_0. Therefore,$\dim H = TC_{d+2}^2 - \mathrm{rank}(H_0)$, where $\mathrm{rank}(H_0)$ denote the rank of H_0. According to Theorem 4, $\dim W_{2\mu+1}^{\mu}(I_1\Delta) = TC_{2\mu+3}^2 - E_I C_{\mu+2}^2$. Hence, when $d = 2\mu+1$, the matrix H_0 is row full rank. So, for $d \geq 2\mu+1$, the H_0 must be row full rank. So,for $d \geq 2\mu+1$, $\dim W_d^{\mu}(I_1\Delta) = TC_{d+2}^2 - E_I C_{\mu+2}^2$. □

Remark When $d \leq 2\mu$, it is difficult for obtaining the $\dim W_d^{\mu}(I_1\Delta)$. In [8], the $\dim W_2^1(St(V))$ is presented. Using B-net method, the $\dim W_2^1(I_1\Delta_{mn}^1)$ is also presented, where Δ_{mn}^1 denotes type-1 triangulation.

We are interested in whether $\dim W_2^1(I_1\Delta)$ is instability, i.e. whether $\dim W_2^1(I_1\Delta)$ not only depends on the topology structure of Δ but also depends on the geometry property of Δ and appointed points.

4. Local supported bases for $W_3^1(I_1\Delta)$

In the section, we will consider the local supported bases for $W_3^1(I_1\Delta)$, where Δ is any triangulation. In fig.1, four local supported multivariate weak functions are shown. From left to right, these functions are called B^1, B^2, B^3, B^4 respectively. In Fig.1, we give the values of $B^i, i = 1,2,3,4$ at the geometric centers of the triangles (these values are placed inside the triangles) and also the values $B^i, D_x B^i, D_y B^i, i = 2,3,4$, respectively (given by the triples $(\cdot,\cdot,\cdot)$), at the appointed point lying on the interior grid edge. These values completely determine B^i with the exception of a translation.

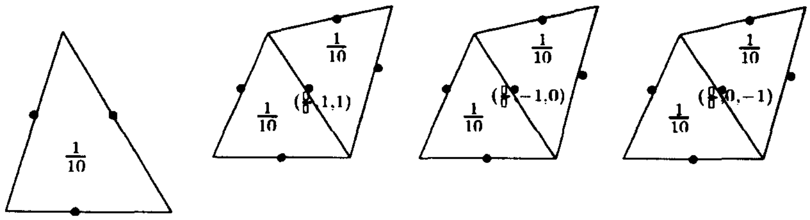

Fig.1

By using the values in Fig.1, we can prove B^i and their translations are line independent. Moreover, we have

Theorem 6 *The $B^i, i = 1,2,3,4$ and their translations are local supported bases of* $W_3^1(I_1\Delta)$.

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多元弱样条函数空间的维数

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摘 要: 讨论了多元弱样条一点处的维数公式及任意三角剖分下的维数公式. 得到了 1- 型剖分下 $W_3^1(I_1\Delta_{mn}^1)$ 的维数与局部支集样条基.