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Uniform Convergence of Higher Order Fejér Interpolation Polynomials in a Complex Domain *

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Abstract: For a Jordan domain D in the complex plane satisfying certain boundary conditions a function $f \in B(\overline{D})$, we prove that the corresponding higher order Fejér interpolation polynomials based on Fejér points converge to $f(z)$ uniformly on $\overline{D}$. These extend some known results.

Key words: complex domain; higher order Fejer interpolation; uniform convergence.

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1. Introduction

Let D be a Jordan domian in the complex plane, $\partial D = \Gamma$. A function $z = \Psi(w)$ conformally maps $|w| > 1$ onto $C\backslash\overline{D}$ so that $\Psi(\infty) = \infty$, $\Psi'(\infty) > 0$. Without loss of generality we may assume $\Psi'(\infty) = 1$. Denote by $A(\overline{D})$ the functions analytic in D and continuous on $\overline{D} = D\bigcup\Gamma$. It is well known that for arbitrary system of nodes on unit circle there exists a function $f \in A(|z| \leq 1)$ such that the corresponding Lagrange interpolation polynomials do not converge uniformly on $|z| \leq 1$[1]. In 1965 Curtiss[2] considered the subclass $B(\overline{D})$ of $A(\overline{D})$ i.e., $f(z) \subset A(\overline{D})$ and $f(z)$ has bounded variation on Γ. He obtained

Theorem A *Let D be a Jordan domain enclosed by an analytic curve and $f \in B(\overline{D})$. Then for Fejér points one has*

$$\lim_{n\to\infty} \|\mathrm{ln}(f,z) - f(z)\|_\infty = 0,$$

where $\|f\|_\infty = \max_{z\in\overline{D}}|f(z)|$, $L_n(f,z)$ *is Lagrange interpolation.*

In 1969 Al'per and Kalinogorskaja[3] improved the boundary condition of D in Theorem A, and obtained.

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Theorem B *Suppose that $D \in \Delta''_\alpha$, i.e., $\Psi'(w)$ is continuous and does not vanish on $|w| \geq 1$, and $\Psi''(w)$ is continuous and belongs to Lipα, $0 < \alpha \leq 1$, on $|w| \geq 1$. Suppose $f \in B(\overline{D})$, then for Fejér points*

$$\lim_{n \to \infty} \|L_n(f,z) - f(z)\|_\infty = 0.$$

Consider Fejér points and non-negative integer q, and denote

$$A_{k,j}(z) = [\frac{\omega(z)}{z - z_k}]^{q+1} \frac{(z-z_k)^j}{j!} \sum_{\nu=0}^{q-j} \alpha_{k,\nu}(z - z_k)^\nu, j = 0, 1, \ldots, q; k = 1, 2, \ldots, n. \quad (1.1)$$

where $\omega(z) = \prod_{k=1}^{n}(z - z_k)$, and $\alpha_{k,\nu}(q,n)$ is defined as

$$(\frac{z - z_k}{\omega(z)})^{q+1} = \sum_{\nu=0}^{\infty} \alpha_{k,\nu}(z - z_k)^\nu. \quad (1.2)$$

The polynomial of degree $N := (q+1)n - 1$

$$H_N(f,z) = \sum_{k=1}^{n} \sum_{j=0}^{q} A_{k,j}(z) f^{(j)}(z_k) \quad (1.3)$$

is called Hermite interpolation polynomial which satisfies interpolation conditions

$$H_N^{(j)}(f, z_k) = f^{(j)}(z_k), j = 0, 1, 2, \ldots, q; k = 1, 2, \ldots, n. \quad (1.4)$$

It is well known that when $m \leq N$, $H_N(p_m, z) = p_m(z)$, where $p_m(z)$ is a polynomial of degree m.

For arbitrary sequence of numbers $\{a_k^{(j)}\}, k = 1, 2, \ldots, n; j = 1, 2, \ldots, q$, the interpolation polynomial of degree $N := (q+1)n - 1$

$$\tilde{H}_N(f,z) = \sum_{k=1}^{n} A_{k,0}(z) f(z_k) + \sum_{k=1}^{n} \sum_{j=1}^{q} A_{k,j}(z) a_k^{(j)} \quad (1.5)$$

satisfies the interpolation conditions

$$\tilde{H}_N(f, z_k) = f(z_k), k = 1, 2, \ldots, n, \quad (1.6)$$

$$\tilde{H}_N^{(j)}(f, z_k) = a_k^{(j)}, k = 1, 2, \ldots, n; j = 1, 2, \ldots, q. \quad (1.7)$$

If $a_k^{(j)} = 0$ in (1.5) and (1.7), then there is Fejér interpolation polynomial

$$F_N(f,z) = \sum_{k=1}^{n} A_{k,0}(z) f(z_k) \quad (1.8)$$

which satisfies conditions

$$F_N(f,z_k) = f(z_k), k = 1,2,\ldots,n, \tag{1.9}$$

$$F_N^{(j)}(f,z_k) = 0, k = 1,2,\ldots,n; j = 1,2,\ldots,q. \tag{1.10}$$

For (1.8), if $q \geq 2$, then we call it higher order Fejér interpolation polynomial. In (1.3),(1.5) and (1.8) set $q = 0$,we obtain Lagrange interpolation polynomial of degree $n-1$.

In 1939[4] Lozinskii proved that for nodes system $\{e^{\frac{2k\pi i}{n}}\}_{k=1}^{n}$, there exists a function $f_0 \in A(|z| \leq 1)$ such that

$$\overline{\lim_{n\to\infty}} F_{2n-1}(f_0, 1) = \infty,$$

that is to say, when $q = 1$, the corresponding Fejér interpolation polynomial does not converge uniformly to $f_0(z)$ on $|z| \leq 1$. It is natural to raise the following problem: for Fejér points and $f \in B(\overline{D})$, does Fejér interpolation polynomial converge uniformly to $f(z)$ on $\overline{D}$? If it is right, what is the boundary condition of D?

Denote by $\mu(t)$ the uniform module of continuity of function $\arg\psi'(e^{i\theta})$. If

$$\int_0^a \frac{\mu(t)}{t}|\ln t|\,\mathrm{d}t < +\infty, a > 0,$$

then we say $D \in J_0$. Similarly, denote by $\mu_1(t)$ the uniform module of continuity of function $\psi'(e^{i\theta})$ on $|w| = 1$. If

$$\int_0^a \frac{\mu_1(t)}{t}|\ln t|^2\,\mathrm{d}t < +\infty, a > 0,$$

then we say $D \in J_1$. Denote by $\mu_2(t)$ the uniform module of continuity of $\psi''(s^{i\theta})$ on $|w| = 1$, if

$$\int_0^a \frac{\mu_2(t)}{t}|\ln t|\,\mathrm{d}t < +\infty, a > 0,$$

then we say $D \in J_2$[5]. It is clear that $J_0 \supset J_1 \supset J_2$. In this paper, the following theorems are obtained.

Theorem 1 *Suppose $D \in J_2$ and $f \in B(\overline{D})$, then for Fejér points*

$$\lim_{n\to\infty} \|f(z) - F_N(f,z)\|_\infty = 0, \ q \text{ is a non-negative integer.}$$

Theorem 2 *Suppose $D \in J_2$ and $f \in B(\overline{D})$, then for Fejér points and arbitrary non-negative integer q, under the additional condition*

$$\max_{0\leq k\leq n} |a_k^{(j)}| = o(\frac{n^j}{\ln n}), j = 1,2,\cdots,q, \tag{1.11}$$

one has

$$\lim_{n\to\infty} \|H_N(f,z) - f(z)\|_\infty = 0.$$

Since that Lagrange interpolation is the particular case of interpolation (1.3), (1.5) or (1.8) and that $\Delta''_\alpha \in J_2$, Theorem 1 and Theorem 2 extend Theorem A and Theorem B, as well as the recent work [6] in which the domain D is unit disk.

2. Some Lemmas

Lemma 1 *Suppose $D \in J_0$, then*

1^0 $\omega(z) = \prod_{k=1}^{n}(z - z_k)$ *is bounded on* D (2.1)

$0 < A_2 n \le \omega'(z_k) \le A_1 n$, A_1, A_2 *are constants*, $k = 1, 2, \cdots, n$. (2.1)′

$$2^0 \sum_{k=1}^{n} |l_k(z)|^{1+\delta} = \begin{cases} O(\ln n), & \delta = 0 \quad (2.2) \\ O(1), & \delta > 0 \quad (2.2)' \end{cases}$$

where

$$l_k(z) = \frac{\omega(z)}{(z - z_k)\omega'(z_k)} = \prod_{l=1, l\neq k}^{n} \frac{z - z_l}{z_k - z_l}.$$

$$3^0 \max_{z\in\overline{D}} \sum_{k=1}^{n} |A_{k,j}(z)| = O\left(\frac{\ln n}{n^j}\right), j = 1, 2, \ldots, q. \quad (2.3)$$

The conclusions 2^0 and 3^0 had been obtained under the condition J_1 or J_2 respectively in [5]. However, all the conclusions $1^0 \sim 3^0$ can be obtained[7] under only the condition j_0.

Lemma 2 *Suppose $D \in J_2, z_k (k = 1, 2, \ldots, n)$ is Fejér point, then*

$$\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z - z_k}{\omega(z)}\right)\Big|_{z=z_k} = -\frac{1}{2} + O\left(\frac{\ln n}{n}\right), \quad k = 1, 2, \ldots, n.$$

Proof The function $\frac{z - z_k}{\omega(z)}$ is analytic in the neighborhood of z_k.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z - z_k}{\omega(z)}\right)\Big|_{z=z_k} &= \lim_{z\to z_k} \frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{z - z_k}{\omega(z)}\right) \\ &= \lim_{z\to z_k} \frac{\omega(z) - (z - z_k)\omega'(z)}{\omega^2(z)} \\ &= \lim_{z\to z_k} \frac{\omega'(z) - \omega'(z) - (z - z_k)\omega''(z)}{2\omega(z)\omega'(z)} \\ &= -\frac{\omega''(z_k)}{2\omega'(z_k)^2}. \end{aligned}$$

by the known result[8]:

$$\frac{\omega''(z_k)}{\omega'(z_k)^2} = 1 + O\left(\frac{\ln n}{n}\right),$$

Lemma 2 is proved.

Lemma 3 *Suppose $D \in J_2$ and $\{z_k\}_{k=1}^{n}$ are Fejér points. Then*

$$a_{k,0} = \frac{1}{\omega'(z_k)^{q+1}},$$

$$a_{k,1} = (q+1)\frac{1}{\omega'(z_k)^q}[-\frac{1}{2} + O(\frac{\ln n}{n})],$$

for $2 \le \nu \le q$, *we have*

$$a_{k,\nu} = \frac{(q+1)!}{\nu!(q+1-\nu)!}\frac{1}{\omega'(z_k)^{q+1-\nu}}[(-\frac{1}{2})^\nu + O(\frac{\ln n}{n})] + \frac{1}{\omega'(z_k)^{q+2-\nu}}\varphi_{k,\nu}(z_k),$$

where $\varphi_{k,\nu}(z_k)$ *is a constant.*

Proof The functions $\frac{z-z_k}{\omega(z)}$ and $(\frac{z-z_k}{\omega(z)})^{q+1}$ are both analytic in the neighborhood of z_k, and so are their derivatives. According to the definition (1.2), we have

$$\begin{aligned} a_{k,0} &= (\frac{z-z_k}{\omega(z)})^{q+1}\Big|_{z=z_k} = \frac{1}{\omega'(z_k)^{q+1}}, \\ a_{k,1} &= \frac{\mathrm{d}}{\mathrm{d}z}(\frac{z-z_k}{\omega(z)})^{q+1}\Big|_{z_k} = (q+1)[(\frac{z-z_k}{\omega(z)})^q(\frac{z-z_k}{\omega(z)})']_{z_k} \\ &= (q+1)\frac{1}{\omega'(z_k)^q}[-\frac{1}{2} + O(\frac{\ln n}{n})], (\text{Lemma2}) \\ a_{k,2} &= \frac{1}{2}[(\frac{z-z_k}{\omega(z)})^{q+1}]''_{z_k} \\ &= \frac{(q+1)}{2!}\{q(\frac{z-z_k}{\omega(z)})^{q-1}[(\frac{z-z_k}{\omega(z)})']^2 + (\frac{z-z_k}{\omega(z)})^q(\frac{z-z_k}{\omega(z)})''\}_{z_k} \\ &= \frac{(q+1)!}{2!(q+1-2)!}\{(\frac{z-z_k}{\omega(z)})^{q-1}[(\frac{z-z_k}{\omega(z)})']^2\}_{z_k} + [(\frac{z-z_k}{\omega(z)})^q\varphi_{k,2}(z)]_{z_k} \end{aligned}$$

where $\varphi_{k,2}(z) = \frac{(q+1)!}{2!(q+1-2)!}(\frac{z-z_k}{\omega(z)})''$ is analytic in the neighborhood of z_k, consequently, $\varphi_{k,2}(z_k)$ is a constant. Thus, applying Lemma 2, we have

$$a_{k,2} = \frac{(q+1)!}{2!(q+1-2)!}\frac{1}{\omega'(z)^{q-1}}[(-\frac{1}{2})^2 + O(\frac{\ln n}{n})] + \frac{1}{\omega'(z_k)^q}\varphi_{k,2}(z_k).$$

Similarly,

$$\begin{aligned} a_{k,3} &= \frac{(q+1)!}{3!(q+1-3)!}\{(\frac{z-z_k}{\omega(z)})^{q-2}[(\frac{z-z_k}{\omega(z)})']^3 + 3q(\frac{z-z_k}{\omega(z)})^{q-1}(\frac{z-z_k}{\omega(z)})'(\frac{z-z_k}{\omega(z)})'' + \\ &\quad (\frac{z-z_k}{\omega(z)})^q(\frac{z-z_k}{\omega(z)})'''\}_{z_k} \\ &= \frac{(q+1)!}{3!(q+1-3)!}\{(\frac{z-z_k}{\omega(z)})^{q-2}_{z_k}[(\frac{z-z_k}{\omega(z)})'_{z_k}]^3\} + [(\frac{z-z_k}{\omega(z)})^{q-1}\varphi_{k,3}(z)]_{z_k} \\ &= \frac{(q+1)!}{3!(q+1-3)!}\frac{1}{\omega'(z_k)^{q-2}}[(-\frac{1}{2})^3 + O(\frac{\ln n}{n})] + \frac{1}{\omega'(z_k)^{q-1}}\varphi_{k,3}(z_k), \end{aligned}$$

where $\varphi_{k,3}(z) = \frac{(q+1)!}{3!(q+1-3)!}[3q(\frac{z-z_k}{\omega(z)})'(\frac{z-z_k}{\omega(z)})'' + (\frac{z-z_k}{\omega(z)})(\frac{z-z_k}{\omega(z)})''']$ is analytic in the neighborhood of z_k.

For general $\nu, 2 \le \nu \le q$, we have

$$a_{k,\nu} = \frac{(q+1)!}{\nu!(q+1-\nu)!}\{(\frac{z-z_k}{\omega(z)})'\}]''\}_{z_k} + [(\frac{z-z_k}{\omega(z)})^{q+1-(\nu-1)}\varphi_{k,\nu}(z)]_{z_k}$$
$$= \frac{(q+1)!}{\nu!(q+1-\nu)!}\frac{1}{\omega'(z_k)^{k+1-\nu}}[(\frac{z-z_k}{\omega(z)})'_{z_k}]^{\nu} + \frac{1}{\omega'(z_k)^{q+2-\nu}}\varphi_{k,\nu}(z_k),$$

where $\varphi_{k,\nu}(z)$ is analytic in the neighborhood of z_k. By (2.1)′ and Lemma 2, the proof of Lemma 3 is completed.

Lemma 4 *Suppose that $\psi''(w)$ is continuous on $|w| \ge 1$ and $f \in B(\overline{D})$. Then the Lagrange interpolation polynomial based on Fejér points*

$$L_n(f,z) = \sum_{k=1}^{n} l_k(z)f(z_k)$$

uniformly converges to $f(z)$ on $\overline{D}$, i.e.

$$\lim_{n\to\infty} \|L_n(f,z) - f(z)\|_\infty = 0.$$

Proof Step 1. Since $\psi''(w)$ continues on $|w| \ge 1$, it follows that

$$\psi(w) = \psi(u) + \psi'(u)(w-u) + \int_u^w \psi''(t)(w-t)\mathrm{d}t \tag{2.4}$$

where the intergration path is along the straight line connecting $w(|w|=1)$ with $u(|u|>1)$ if $\overline{wu}$ does not intersect circle $|w|=1$, otherwise, it is along the circular arc and exterior part of $\overline{wu}$. From (2.4) one has

$$|\psi(w) - \psi(u) - \psi'(u)(w-u)| = O(|w-u|^2).$$

It follows from Lemma 1 that

$$|\frac{1}{w-u} - \frac{\psi'(u)}{\psi(w)-\psi(u)}| \le B_0,\ B_0 --- \text{const.} \tag{2.5}$$

Step 2. It is well known that for $f \in B(\overline{D})$, there is a polynomial $P(z)$ such that

$$|f(z) - P(z)| < \varepsilon.$$

Let $\triangle(z) = f(z) - P(z)$, so $\triangle(z) \in B(\overline{D})$, and

$$|\triangle(z)| < \varepsilon. \tag{2.6}$$

Consider the Faber transformation

$$\frac{1}{2\pi i}\int_{|w|=1} \frac{\triangle[\psi(u)]}{u-w}\mathrm{d}u$$

which defines two functions $\varphi_1(w)$ and $\varphi_2(w)$ analytic in $|w| < 1$ and $|w| > 1$ respectively. Under the assumption that $\psi''(w)$ continues on $|w| \geq 1$, one has[9]

1^0 $\varphi_1(w)$ and $\varphi_2(w)$ are continuous on $|w| \leq 1$ and $|w| \geq 1$ respectively, moreover

$$\|\varphi_1(w)\|_\infty = O(1)\|\triangle\|_\infty, \quad \|\varphi_2(w)\|_\infty = O(1)\|\triangle\|_\infty; \tag{2.7}$$

2^0 $\varphi_1(w) \in B(|w| \leq 1), \varphi_2(w) \in B(|w| \geq 1);$ (2.8)

3^0 $\triangle[\psi(w)] = \varphi_1(w) - \varphi_2(w), \ |w| = 1.$ (2.9)

Step 3. Suppose $z \in \Gamma$, and let n be large enough,

$$\begin{aligned} f(z) - L_n(f,z) =& P(z) + \triangle(z) - L_n(P,z) - L_n(\triangle,z) \\ =& \triangle(z) - L_n(\triangle,z) \\ =& \triangle(z) - \sum_{k=1}^{n} \triangle(z_k) \frac{\omega(z)}{\omega'(z_k)(z-z_k)} \\ =& \triangle[\psi(w)] - \sum_{k=1}^{n} \triangle[\psi(w_k)] \frac{(w^n-1)\prod_n(w)}{\psi(w)-\psi(w_k)} \frac{w_k\psi'(w_k)}{n\prod_n(w_k)} \\ =& \triangle[\psi(w)] - \sum_{k=1}^{n} \triangle[\psi(w_k)] \frac{w^n-1}{n(w-w_k)} \\ & - \frac{w^n-1}{n} \sum_{k=1}^{n} \triangle[\psi(w_k)] w_k [\frac{\psi'(w_k)}{\psi(w)-\psi(w_k)} - \frac{1}{w-w_k}] \\ & - \frac{1}{n} \sum_{k=1} n\triangle[\psi(w_k)] \frac{w_k\psi'(w_k)(w_n-1)}{\psi(w)-\psi(w_k)} [\frac{\prod_n(w)}{\prod_n(w_k)} - 1] \end{aligned} \tag{2.10}$$

where $\prod_n(w) = \prod_{k=1}^{n} \frac{z-z_k}{w-w_k}$. For brevity let $\triangle[\psi(w)] = \triangle(w)$. Then

$$f(z) - L_n(f,z) = \varphi_1(w) - \varphi_2(w) - L_n(\varphi_1,w) + L_n(\varphi_2,w) + B(\triangle,w) + C(\triangle,w) \tag{2.11}$$

where $B(\triangle,w)$ and $C(\triangle,w)$ denote the last two terms in (2.10) respectively. We are going to estimate every term in the right hand side of equality (2.11).

Since $\varphi_1 \in B(|w| \leq 1)$, by virtue of Curtiss theorem[2], for $|w| = 1$ one has

$$\lim_{n\to\infty} \|\varphi_1(w) - L_n(\varphi_1,w)\|_\infty = 0. \tag{2.12}$$

By virtue of (2.7)

$$\|\varphi_2\|_\infty = O(1)\varepsilon. \tag{2.13}$$

By virtue of (2.5),

$$\|B(\triangle,w)\|_\infty = O(1)\varepsilon. \tag{2.14}$$

Since $J_1 \supset J_2$, it follows that[5]

$$\left|\frac{\prod_n(w)}{\prod_n(w_k)} - 1\right| = o(\frac{1}{\ln n}),$$

and so

$$\|C(\triangle, w)\|_\infty = o(1)\,O(\ln n)\,o(\frac{1}{\ln n}) = o(1). \tag{2.15}$$

It remains to prove that

$$\|L_n(\varphi_2, w)\|_\infty = O(\varepsilon).$$

By similar argument in [3.p.22], Lemma 4 is proved.

Lemma 4 has improved the boundary condition of D in Theorem A and Theorem B.

3. The proofs of Theorem 1 and Theorem 2

Proof of Theorem 1 In Section 2 we have mentioned that for any $\varepsilon > 0$, there is a polynomial such that

$$|f(z) - P(z)| < \varepsilon.$$

Note $\triangle(z) = f(z) - P(z)$, then $\triangle(z) \in B(\overline{D})$ provided $f \in B(\overline{D}$, and

$$|\triangle(z)| < \varepsilon. \tag{3.1}$$

Taking n large enough, from Section 1 we have $H_N(p, z) = P(z)$, thus

$$\begin{aligned} F_N(f,z) - f(z) =& F_N(f,z) - F_N(P,z) + F_N(P,z) - H_N(P,z) + P(z) - f(z) \\ =& F_N(\triangle, z) - \sum_{k=1}^{n}\sum_{j=1}^{q} A_{k,j}(z)P^{(j)}(z_k) - \triangle(z) \\ =&: I - II - III. \end{aligned} \tag{3.2}$$

Since $p(z)$ is independent of n, $p^{(j)}(z_k)$ is uniformly bounded for k and j, from (2.3) there is

$$\|II\|_\infty \le \max_{z\in\overline{D}} \sum_{j=1}^{q}\sum_{k=1}^{n} |A_{k,j}(z)p^{(j)}(z_k)| = \sum_{j=1}^{q} O(\frac{\ln n}{n^j}) \to 0,\ n \to \infty. \tag{3.3}$$

Clearly, it is enough to estimate I.

$$\begin{aligned} F_N(\triangle, z) &= \sum_{k=1}^{n} A_{k,o}(z)\triangle(z_k) = \sum_{k=1}^{n}(\frac{\omega(z)}{z - z_k})^{q+1}\sum_{\nu=0}^{q}\alpha_{k,\nu}(z - z_k)^\nu \triangle(z_k) \\ &= \sum_{\nu=0}^{q}\sum_{k=1}^{n}\triangle(z_k)\alpha_{k,\nu}\frac{\omega(z)^{q+1}}{(z - z_k)^{q+1-\nu}} =: \sum_{\nu=0}^{q} Q_\nu. \end{aligned} \tag{3.4}$$

We show that $\|Q_\nu\|_\infty \to 0$, $n \to \infty$, $\nu = 0, 1, 2, \cdots, q$.

Set $\nu = 0$ while $q \ge 0$. By Lemma 3

$$\begin{aligned} Q_0 &= \sum_{k=1}^{n}\alpha_{k,0}(\frac{\omega(z)}{z - z_k})^{q+1}\triangle(z_k) = \sum_{k=1}^{n}\frac{1}{\omega'(z_k)^{q+1}}l_k(z)^{q+1}\omega'(z_k)^{q+1}\triangle(z_k) \\ &= \sum_{k=1}^{n} l_k(z)^{q+1}\triangle(z). \end{aligned} \tag{3.5}$$

If $q=0$, rewrite (3.5) as

$$Q_0=\sum_{k=1}^{n} l_k(z)^{q+1}\triangle(z)=L_n(\triangle,z)-\triangle(z)+\triangle(z). \tag{3.6}$$

Since $\triangle(z)\in B(\overline{D})$, from Lemma 4

$$||Q_0||_\infty\le 2\varepsilon,\ \text{ when } n>n_0.$$

If $q\ge 1$, it follows from (3.5) and (2.2)$'$, that $||Q_0||<\varepsilon$. Thus when $q\ge 0$, one has

$$||Q_0||_\infty\to 0,\ \text{ when } n\to\infty. \tag{3.7}$$

Set $\nu=1$ while $q\ge 1$, by applying Lemma 3, one has

$$\begin{aligned}
Q_1 &=\sum_{k=1}^{n}\alpha_{k,1}\frac{\omega(z)^{q+1}}{(z-z_k)^q}\triangle(z_k)\\
&=\sum_{k=1}^{n}(q+1)\frac{1}{\omega'(z_k)^q}[-\frac{1}{2}+O(\frac{\ln n}{n})]l_k(z)^q\omega'(z_k)^q\omega(z)\triangle(z_k)\\
&=(q+1)\omega(z)\sum_{k=1}^{n}(-\frac{1}{2})l_k(z)^q\triangle(z_k)+(q+1)\omega(z)\sum_{k=1}^{n}l_k(z)^q\triangle(z)O(\frac{\ln n}{n})\\
&=:A+B.
\end{aligned}$$

We estimate A as follows. When $q=1$, write

$$\begin{aligned}
A &=(q+1)\omega(z)\sum_{k=1}^{n}(-\frac{1}{2})l_k(z)^q\triangle(z_k)\\
&=-\frac{q+1}{2}\omega(z)[L_n(\triangle,z)-\triangle(z)+\triangle(z)].
\end{aligned}$$

From (2.1) and Lemma 4, one has

$$||A||_\infty=O(1)\varepsilon,\ \text{ when } n>n_1.$$

If $q\ge 2$, from (2.2) and (2.1),

$$||A||_\infty=||(q+1)(-\frac{1}{2})\omega(z)\sum_{k=1}^{n}l_k(z)^q\triangle(z_k)||_\infty=O(\varepsilon).$$

When $q=1$, we have

$$||B||_\infty=O(\frac{\ln n}{n})O(\ln n)\varepsilon=\varepsilon O(\frac{\ln^2 n}{n});$$

when $q\ge 2$,

$$||B||_\infty=O(1)\varepsilon O(\frac{\ln n}{n})=\varepsilon O(\frac{\ln n}{n}).$$

Thus, when $q \geq 1$, we have

$$\|Q_1\|_\infty \to 0, n \to \infty. \tag{3.8}$$

Similarly, consider ν: $2 \leq \nu \leq q$

$$\begin{aligned}
Q_\nu =& \sum_{k=1}^{n} \alpha_{k,\nu} \frac{\omega(z)^{q+1}}{(z-z_k)^{k+1-\nu}} \triangle(z_k) \\
=& \sum_{k=1}^{n} \{ \frac{(q+1)!}{\nu!(q+1-\nu)!} \frac{1}{\omega'(z_k)^{k+1-\nu}} [(-\frac{1}{2})^\nu + O(\frac{\ln n}{n})] + \\
& \frac{1}{\omega'(z_k)^{q+2-\nu}} \varphi_{k,\nu}(z_k) \} \omega(z)^\nu l_k(z)^{q+1-\nu} \omega'(z-k)^{q+1-\nu} \triangle(z_k) \\
=& \omega(z)^\nu \frac{(q+1)!}{\nu!(q+1-\nu)!} \sum_{k=1}^{n} \frac{1}{\omega'(z_k)^{q+1-\nu}} (-\frac{1}{2})^\nu l_k(z)^{q+1-\nu} \omega'(z)^{q+1-\nu} \triangle(z_k) + \\
& \omega^\nu \frac{(q+1)!}{\nu!(q+1-\nu)!} \sum_{k=1}^{n} \frac{1}{\omega'(z_k)^{q+1-\nu}} l_k(z)^{q+1-\nu} \omega'(z_k)^{q+1-\nu} \triangle(z_k) O(\frac{\ln n}{n}) + \\
& \omega(z)^\nu \sum_{k=1}^{n} \frac{1}{\omega'(z_k)^{q+2-\nu}} \varphi_{k,\nu}(z_k) l_k(z)^{q+1-\nu} \omega'(z_k)^{q+1-\nu} \triangle(z_k) \\
=:& A_\nu + B_\nu + C_\nu.
\end{aligned}$$

Consider

$$A_\nu = (-\frac{1}{2})^\nu \omega(z)^\nu \frac{q+1)!}{\nu!(q+1-\nu)!} \sum_{k=1}^{n} l_k(z)^{q+1-\nu} \triangle(z_k),$$

When $q+1-\nu = 1$, from (2.1) and Lemma 4

$$\begin{aligned}
\|A_\nu\|_\infty =& \|(-\frac{1}{2})^\nu \frac{(q+1)!}{\nu!(q+1-\nu)!} [L_n(\triangle, z) - \triangle(z) + \triangle(z)]\|_\infty \\
=& O(\varepsilon), \ n > n_2.
\end{aligned}$$

When $q+1-\nu \geq 2$, from (2.1) and (2.2)$'$,

$$\|A_\nu\|_\infty = O(1)\varepsilon.$$

Consider

$$B_\nu = \omega(z)^\nu \frac{(q+1)!}{\nu!(q+1-\nu)!} \sum_{k=1}^{n} l_k(z)^{q+1-\nu} \triangle(z_k) O(\frac{\ln n}{n}),$$

$$\|B_\nu\|_\infty = \begin{cases} O(\frac{\ln n}{n}) \|L_n(\triangle, z) - \triangle(z) + \triangle(z)\|_\infty = \varepsilon O(\frac{\ln n}{n}), & q+1-\nu = 1, \\ O(1)\varepsilon O(\frac{\ln n}{n}) = \varepsilon O(\frac{\ln n}{n}), & q+1-\nu \geq 2. \end{cases}$$

Finally,

$$C_\nu = \omega^\nu \sum_{k=1}^{n} \frac{1}{\omega'(z_k)} \varphi_{k,\nu}(z_k) l_k(z)^{q+1-\nu} \triangle(z_k),$$

$$\|C_\nu\|_\infty = \begin{cases} O(\frac{1}{n})O(\ln n)\varepsilon = \varepsilon O(\frac{\ln n}{n}), & q+1-\nu=1, \\ O(\frac{1}{n})\varepsilon, & q+1-\nu \geq 2. \end{cases}$$

By summarizing the above argument, it follows that

$$\|Q_\nu\|_\infty \to 0, n \to \infty, 2 \leq \nu \leq q. \tag{3.9}$$

Combining (3.4) with (3.7), (3.8) and (3.9), we have

$$\|F_N(\triangle, z)\|_\infty \to 0,\ n \to \infty. \tag{3.10}$$

From (3.1), (3.2), (3.3) and (3.10), Theorem 1 is proved.

Proof of Theorem 2 Suppose $P(z)$ satisfies the conditions in Theorem 1.

$$\begin{aligned} f(z) - \tilde{H}_N(f,z) =& f(z) - P(z) + P(z) - \tilde{H}_N(f,z) \\ =& \triangle(z) - F_N(\triangle, z) + \sum_{k=1}^{n}\sum_{j=1}^{q} A_{k,j}(z)P^{(j)}(z_k) - \\ & \sum_{k=1}^{n}\sum_{j=1}^{q} A_{k,j}(z)\alpha_k^{(j)}. \end{aligned} \tag{3.11}$$

Since $P^{(j)}(z_k)$ are uniformly bounded with respect to k and j, it follows from (2.3) that

$$\|\sum_{k=1}^{n}\sum_{j=1}^{q} A_{k,j}(z)P^{(j)}(z_k)\|_\infty = \sum_{j=1}^{q} O(\frac{\ln n}{n}) \to 0,\ n \to 0.$$

From (2.3) and (1.1),

$$\|\sum_{k=1}^{n}\sum_{j=1}^{q} A_{k,j}(z)\alpha_k^{(j)}\|_\infty = o(1), n \to \infty.$$

By considering (3.1) and (3.10), Theorem 2 is proved.

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复域中高阶 Fejér 插值的一致收敛性

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摘 要: 对于复域中满足某种条件的 Jordan 区域 D 和函数 $f \in B(\overline{D})$, 证明了基于 Fejér 点的高阶 Fejér 插值多项式一致收敛于对应的函数 $f(z)$ 于 $\overline{D}$ 上. 本文中的这些定理推广了某些已知的结果.