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The Second Separation Theorem in Locally β-Convex Spaces and the Boundedness Theorem in Its Conjugate Cones *

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Abstract: This paper deals with the locally β-convex analysis that generalizes the locally convex analysis. The second separation theorem in locally β-convex spaces, the Minkowski theorem and the Krein-Milman theorem in the β-convex analysis are given. Moreover, it is obtained that the UF-boundedness and the UB-boundedness in its conjugate cone are equivalent if and only if X is subcomplete.

Key words: locally β-convex space; β-subseminorm; β-extreme point(set); β-Minkowski functional; conjugate (topological) cone; subcomplete; $UF-(UB\text{-})$boundedness.

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1. Introduction

Let $0 < \beta \leq 1$ be a fixed number, let X be a vector space and A a subset of X. A is called β-convex if for every $x, y \in A$ we have $[x, y]_\beta \subset A$. Where

$$[x, y]_\beta := \{\lambda x + \mu y : \lambda, \mu \geq 0, \lambda^\beta + \mu^\beta = 1\};$$

$$(x, y)_\beta := [x, y]_\beta \backslash \{x, y\}.$$

In this paper we use $\mathrm{co}_\beta A$ and $\mathrm{aco}_\beta A$ to denote the β-convex hull and the absolutely β-convex hull of A respectively, θ the neutral element of spaces.

Definition 1.1 *Let $B \subset A$ be two subsets of a vector space X. If for every $x, y \in A$, whenever $(x, y)_\beta \bigcap B \neq \Phi$ we have $x, y \in B$, then B is called a β-extremal subset of A. x_0 is called a β-extreme point of A if the singleton $\{x_0\}$ is a β-extremal subset of it. The set of all β-extreme points of A is called the β-extreme points set of A, and is denoted by* $\mathrm{ext}_\beta A$.

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Let A be a β-convex set, a real-valued functional f defined on A is called β-convex if for every $x, y \in A$ we have

$$f(\lambda x + \mu y) \le \lambda^\beta f(x) + \mu^\beta f(y), \lambda, \mu \ge 0, \lambda^\beta + \mu^\beta = 1.$$

The set of all such functionals is denoted by $\text{conv}_\beta(A)$.

A nonnegative, subadditive and β-positively homogeneous (respectively, β-absolutely homogeneous) functional f defined on a vector space X is called a β-subseminorm, snd we use X_β' to denote the family of all such functionals. (respectively, called β-seminorm, and use $X_{a\beta}'$ to denote the family of all such functionals.) When X is a topological vector space, we use X_β^* and $X_{a\beta}^*$ to denote the family of all continuous β-subseminorms and the family of all continuous β-seminorms respectively. The relation

$$X_{a\beta}' \subseteq X_\beta' \subseteq \text{conv}_\beta(X)$$

is clear. If $f \in \text{conv}_\beta(X)(f \in X_{a\beta}')$, then for every $a \in R$, the set

$$B(f, a) := \{x \in X : f(x) \le a\}$$

is β-convex (respectively, absolutely β-convex.)

When A is a star absorbing subset of a vector space X, we call the nonnegative-valued functional

$$P_{A_\beta}(x) := \inf\{t > 0 : x \in t^{\frac{1}{\beta}} A\}, x \in X$$

the β-Minkowski functional generated by A.

Definition 1.2 *Let $0 < \beta \le 1$, a topological vector space X is called locally β-convex if there exists a θ-neighborhood basis of β-convex sets.*

This paper deals with the locally β-convex analysis that makes the locally convex analysis its special case. The β-convexity is far more extensive than the linear convexity (the general convexity). When $0 < \beta < 1$, every nonempty β-convex set is restrained by θ (has θ as one of its limit point), so they have no translation invariant property[2]. [2] studied some algebraic problems in the β-convex analysis, and obtained the first separation theorem. [1] and [6] obtained some fundamental properties about locally β-convex spaces and β-convex functionals, etc.. Until now, however, the theory and the applications of the locally β-convex analysis is far poorer than that of the locally convex analysis. The second separation theorem in locally β-convex spaces X, the Minkowski theorem and the Krein-Milman theorem in the β-convex analysis deduced from the second separation theorem are obtained, and a necessary and sufficient condition for the equivalence of the UF-boundedness and the UB-boundedness in its conjugate cone X_β^* is obtained.

2. The second separation theorem in β-convex spaces and the Minkowski theorem

The following lemmas are the basic facts in the β-convex analysis. Because they are not difficult to verity, we delete their proof.

Lemma 2.1 *Let $0 < \beta \le 1$,and let A be a star absorbing set. Then*

1. *$P_{A_\beta}(x)$ is β-positively homogeneous;*

2. *when A is β-convex , we have $P_{A_\beta} \in X'_\beta$;*

3. *when A is absolutely β-convex , we have $P_{A_\beta} \in X'_{\alpha\beta}$.*

Lemma 2.2 *Let $0 < \beta \le 1$, and let X be a topological vector space. Then*

1. *when A is a star absorbing β-convex subset of X, we have*

$$\mathrm{int}A \subset \{x \in X : P_{A_\beta}(x) < 1\} \subset A \subset \{x \in X : P_{A_\beta}(x) \le 1\} \subset \overline{A};$$

2. *let $f \in X'_\beta$, then $f \in X^*_\beta$ if and only if there is some $a > 0$ such that $\mathrm{int}B(f,a) \neq \Phi$.*

Lemma 2.3 *Let $0 < \beta \le 1$.*

1. *For each locally $\beta-$convex space, there is a θ-neighborhood basis cosisting of (open) $\beta-$barrels (absorbing, closed (respectively,open) and absolutely β-convex set).*

2. *A topological vector space X is locally β-convex if and only if there is a family of continuous β-(sub)seminorms $\{f_\alpha \in X^*_\beta : \alpha \in I\}$ such that $\mathcal{U}(\theta) = \{B(f_\alpha, 1) : \alpha \in I\}$ constitues a θ-neighborhood basis of X.*

3. *A locally β-convex space X is separated if and only if for every $x \neq \theta$, there is some $f \in X^*_{\alpha\beta}$ such that $f(x) > 0$.*

Theorem 2.1 (The second separation theorem) *Let $0 < \beta \le 1$, let X be a locally β-convex space. Let A be a nonempty closed β-convex subset of X and B a nonempty closed subset of X with $A \bigcap B = \Phi$ (for $\beta = 1$, still assume $\theta \in A$). Then if A or B is compact, they can be strongly separated by some continuous β-subseminorm, i.e. there is $f \in X^*_\beta$ such that $\max\{f(x) : x \in A\} < 1 \le \inf\{f(x) : x \in B\}$, if A is compact, and $\sup\{f(x) : x \in A\} \le 1 < \min\{f(x) : x \in B\}$, if B is compact.*

Proof Let $\mathcal{U}(\theta) = \{U_\alpha : \alpha \in I\}$ be a $\theta-$neighborhood basis of X consisting of open absolutely $\beta-$convex sets. Since A or B is compact, we assert that there is $U_{\alpha_0} \in \mathcal{U}(\theta)$ such that $(A + U_{\alpha_0}) \bigcap B = \Phi$, which is equivalent to $(B + U_{\alpha_0}) \bigcap A = \Phi$ as U_{α_0} is circled. Assume to the contrary, that the assertion is wrong, without loss of generality we may assume that A is compact. Then for every $\alpha \in I$, there exist $x_\alpha \in A, y_\alpha \in B$ and $z_\alpha \in U_\alpha$ such that $y_\alpha = x_\alpha + z_\alpha$. It is obvious that $\mathcal{U}(\theta)$ is directed under the set-theoretic relation $\supset$, i.e. for each pair $U_{\alpha_1}, U_{\alpha_2} \in \mathcal{U}(\theta)$, there is $U_{\alpha_3} \in \mathcal{U}(\theta)$ such that $U_{\alpha_1} \bigcap U_{\alpha_2} \supset U_{\alpha_3}$. Now it is obvious that $\{x_\alpha\}, \{y_\alpha\}$ and $\{z_\alpha\}$ turn into three nets and $z_\alpha \longrightarrow \theta$. From the compactness of A, there is some convergent subnet of $\{x_\alpha\}$ .Without loss of generality we may assume that $x_\alpha \longrightarrow x_0$, then $y_\alpha = x_\alpha + z_\alpha \longrightarrow x_0$. Then we have $x_0 \in A \bigcap B$ because A and B are closed. This is contrary to $A \bigcap B = \Phi$. The contradiction means that the assertion holds.

Now let $U_{\alpha_0} \in \mathcal{U}(\theta)$ be such that $(A + U_{\alpha_0}) \bigcap B = \Phi$, and let $U_{\alpha_1} \in \mathcal{U}(\theta)$ such that $U_{\alpha_1} + U_{\alpha_1} \subset U_{\alpha_0}$. By $(A + U_{\alpha_1} + U_{\alpha_1}) \bigcap B = \Phi$ and the fact that U_{α_1} is circled, we have

$$(A + U_{\alpha_1}) \bigcap (B + U_{\alpha_1}) = \Phi \tag{1}$$

and then

$$\overline{(A+U_{\alpha_1})}\bigcap B=\Phi. \tag{2}$$

Since $A+U_{\alpha_1}$ is an open β-convex θ-neighborhood, by lemma 2.1 and lemm2.2 we have $f=P_{(A+U_{\alpha_1})_\beta}\in X^*_\beta$. From (2) we have

$$f(x)<f(y),\forall(x,y)\in A\times B. \tag{3}$$

When A is compact, assume that f takes its maximum at x_0, then

$$\max\{f(x):x\in A\}=f(x_0)<1\leq\inf\{f(x):x\in B\}; \tag{4}$$

When B is compact, we can similarly obtain

$$\sup\{f(x):x\in A\}\leq 1<\min\{f(x):x\in B\}. \tag{5}$$

This completes the proof. □

Corollary *Let X be a locally β-convex Hausdorff space, $x,y\in X$ and $x\neq y$. Then there is $f\in X^*_\beta$ such that $f(x)\neq f(y)$, i.e., X^*_β separates X.*

Proof Since $x\neq y$, we have $y\notin[\theta,x]$ or $x\notin[\theta,y]$. Without loss of generality we may assume that $y\notin[\theta,x]$. For X is separated, $[\theta,x]$ and $\{y\}$ satisfies the condition of theorem 2.1, hence there is $f\in X^*_\beta$ such that $f(x)\neq f(y)$. □

For a nonempty locally convex topological vector space X, it is well-known that its conjugate space X^* is big enough. When X is non-locally convex, it is possible that X^* consists of only one element θ, i.e. $X^*=\{\theta\}$. But if the space X is locally $\beta-$convex, X^*_β constains a great number of non-zero elements. For example, the space $L^\beta[0,1](0<\beta<1)$ is just such a space. When X is locally β-convex space, it is clear that X^*_β is a positive cone, called the conjugate cone of X. In this case, the locally β-convex topology and the conjugate cone X^*_β of X are determined by each other. As the applications of the second separation theorem, we now show the Minkowski theorem and the Krein-Milman theorem in locally $\beta-$convex spaces.

Theorem 2.2 *Let $0<\beta\leq 1$, let X be a locally β-convex Hausdorff space and A a nonempty compact subset of X. Then $\mathrm{ext}_\beta A\neq\Phi$.*

Proof Let $\mathcal{A}$ be the family of all compact $\beta-$extremal subset of A, then $A\in\mathcal{A}$. Under the set-theoretic relation $\supset$, $\mathcal{A}$ is a semi-order set (i.e. for $B_1,B_2\in\mathcal{A}$,$B_1\prec B_2\Longleftrightarrow B_1\supset B_2$) and for each total ordering subset $\mathcal{B}$ of $\mathcal{A}$, $\bigcap\{B:B\in\mathcal{B}\}\in\mathcal{B}$ is its upper bound. Then by the Zorn lemma there is a maximal element $C\in\mathcal{A}$. By the second separation theorem we can show that C is a singleton, so $\Phi\neq C\subset\mathrm{ext}_\beta A$. □

Lemma 2.4 *Let $0<\beta\leq 1$ and $B_1,B_2,\cdots,B_n$ be a finite family of compact β-convex subsets of a topological vector space X. Then $\mathrm{co}_\beta(\bigcup\limits_{i=1}^{n}B_i)$ is compact and β-convex as well.*

In fact, this can be seen from the compactness of $B_1 \times \cdots \times B_n \times A$ and the continuity of $f : B_1 \times \cdots \times B_n \times A \longrightarrow X$:

$$f(x_1, \cdots, x_n, (\lambda_1, \cdots, \lambda_n)) = \sum_{i=1}^{n} \lambda_i x_i.$$

Where $A = \{(\lambda_1, \lambda_2, \cdots, \lambda_n) \in R^n : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i^\beta = 1\}$ is compact in R^n.

Theorem 2.3 *Let $0 < \beta \leq 1$, let X be a locally β-convex Hausdorff space and B a subset of X such that $\overline{\mathrm{co}_\beta B}$ is compact. Then*

$$\mathrm{ext}_\beta(\overline{co_\beta B}) \subset \overline{B} \bigcup \{\theta\}.$$

Proof When $\beta = 1$, from [4] we have $\mathrm{ext}(\overline{\mathrm{co} B}) \subset \overline{B}$, we need only to show the conclusion for $0 < \beta < 1$. As $\overline{\mathrm{co}_\beta B} = \overline{\mathrm{co}_\beta \overline{B}}$, there is no harm in assuming that B is closed, hence it is compact, too. By theorem 2.2 we have $\mathrm{ext}_\beta(\overline{\mathrm{co}_\beta B}) \neq \Phi$. Let $\theta \neq x_0 \in \mathrm{ext}_\beta(\overline{\mathrm{co}_\beta B})$, it is sufficient to show $x_0 \in B$.

Let $\mathcal{U}(\theta)$ be a θ-neighborhood basis of X consisting of β-barrels. For every $U \in \mathcal{U}(\theta)$, by the compactness of B there exists a finite family $x_1, x_2, \cdots, x_n \in B$ such that $B \subset \bigcup_{i=1}^{n} (x_i + U)$. Let

$$B_i = \overline{\mathrm{co}_\beta[([\theta, x_i] + U) \bigcap B]}, (i = 1, 2, \cdots, n).$$

From $B_i \subset \overline{\mathrm{co}_\beta B}$ we know that B_i are compact and β-convex. Then $\mathrm{co}_\beta(\bigcup_{i=1}^{n} B_i)$ is also compact and β-convex by lemma 2.4. From $B \subset \bigcup_{i=1}^{n} B_i$ we have

$$\overline{\mathrm{co}_\beta B} \subset \overline{\mathrm{co}_\beta(\bigcup_{i=1}^{n} B_i)} = \mathrm{co}_\beta(\bigcup_{i=1}^{n} B_i).$$

Then

$$x_0 = \sum_{i=1}^{n} \lambda_i y_i, y_i \in B_i, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i^\beta = 1.$$

From $\theta \neq x_0 \in \mathrm{ext}_\beta(\overline{\mathrm{co}_\beta B})$ there is $1 \leq i_0 \leq n$ such that $\lambda_{i_0} = 1$, and for the other $j \neq i_0$ we have $\lambda_j = 0$. Let $x_0 = y_{i0} \in B_{i_0} \subset [\theta, x_{i_0}] + U$ (notice that the right is closed and β-convex). From the arbitrary property of U we have $x_0 \in \overline{[0,1]B} = [0,1]B$. Because $\theta \neq x_0 \in \mathrm{ext}_\beta(\overline{\mathrm{co}_\beta B})$ we have $x_0 \in B$, this completes the proof. □

It is not difficult to show the following lemmas:

Lemma 2.5 *Let $0 < \beta \leq 1$, let B be a nonempty subset of a topological vector X. Then for every continuous β-convex functional f defined on X we have*

$$\sup\{f(x) : x \in B\} = \sup\{f(x) : x \in \overline{\mathrm{co}_\beta B}\}.$$

Lemma 2.6 *Let $0 < \beta < 1$, let X be a locally β-convex space and A a nonempty compact β-convex subset of X. Let $f \in \mathrm{conv}_\beta(X)$ be an upper semicontinous functional. Then the β-convex programming problem* $\max\{f(x) : x \in A\}$ *has solution in* $\mathrm{ext}_\beta A$.

The following theorem is a generalization of the Minkowski theorem of locally convex analysis, see [4] and [9], etc..

Theorem 2.4 (Minkowski theorem) *Let $0 < \beta < 1$, let X be a locally β-convex Hausdorff space and A a nonempty compact β-convex subset of X. Then for each $B \subset A$, the following conditions are equivalent:*

1. $\overline{\mathrm{co}_\beta B} = A$;
2. $\sup\{f(x) : x \in B\} = \max\{f(x) : x \in A\}, \forall f \in X_\beta^*$;
3. $\mathrm{ext}_\beta A \subset \overline{B} \bigcup \{\theta\}$.

Proof From lemma 2.5 we know that (1) implies (2).

When $x_0 \in A \backslash \overline{\mathrm{co}_\beta B}$, $\{x_0\}$ is closed and $\overline{\mathrm{co}_\beta B}$ is a compact β-convex set. By the second separation theorem there is $f \in X_\beta^*$ such that

$$\sup\{f(x) : x \in B\} = \max\{f(x) : x \in \overline{\mathrm{co}_\beta B}\} < 1 < f(x_0) \leq \max\{f(x) : x \in A\}.$$

This shows that (2) implies (1).

When $\mathrm{ext}_\beta A \subset \overline{B} \bigcup \{\theta\}$, for every $f \in X_\beta^*$, by lemma 2.6 there is $x_0 \in \mathrm{ext}_\beta A \subset \overline{B} \bigcup \{\theta\}$ such that $f(x_0) = \sup\{f(x) : x \in A\}$. As $f(\theta) = 0$, no matter $x_0 = \theta$ or $x_0 \in \overline{B}$, we have

$$\max\{f(x) : x \in A\} = f(x_0) \leq \sup\{f(x) : x \in B\},$$

i.e., (3) implies (2). The fact that (1) implies (3) can be got from theorem 2.3. □

Corollary 1 (Krein-Milman theorem[6]) *Let $0 < \beta \leq 1$, let X be a locally β-convex Hausdorff space and A a nonempty compact β-convex subset of X, then $A = \overline{\mathrm{co}_\beta(\mathrm{ext}_\beta A)}$.*

Corollary 2 *Let $0 < \beta \leq 1$, let X be a locally β-convex Hausdorff space and B a nonempty subset of X such that $\overline{\mathrm{aco}_\beta B}$ is compact, then*

1. $\mathrm{ext}_\beta(\overline{\mathrm{co}_\beta B}) \subset \mathrm{ext}_\beta \overline{B} \bigcup \{\theta\}$;
2. $\mathrm{ext}_\beta(\overline{\mathrm{aco}_\beta B}) \subset \bigcup_{|\lambda|=1} \lambda \mathrm{ext}_\beta \overline{B}$.

3. The equivalence of the UF−boundedness and the UB-boundedness in conjugate cones

When X is a locally β-convex space, by the corollary of the second separation theorem we know that its conjugate cone X_β^* is big enough. In this case, the locally β-convex topology and the conjugate cone X_β^* of X are determined by each other. In this section, we are going to discuss the equivalence problem about two kinds of boundedness under the following two uniformly convergent topologies of X_β^*, i.e. the problem about when the uniform boundedness principle holds in the topological cone X_β^*.

Definition 3.1 *Let E be a convex cone with neutral element θ. E is called a topological cone if there is a topology τ such that the addition and the multiplication with nonnegative numbers are continuous in (E,τ). A set $H \subset (E,\tau)$ is called bounded if H can be absorbed by any θ-neighborhoods, i.e. for every $U \in \mathcal{U}(\theta)$, there is $\lambda_0 > 0$ such that $[0,\lambda_0]H \subset U$.*

Let X be a locally β-convex space, let $\mathcal{B} \subset 2^X$ be a family of subsets of X directed under set-theoretic relation $\subset$. Then X_β^* can be endowed with such a topology that the convergence is equivalent to the uniform convergence on every member of $\mathcal{B}$. This space is denoted by $(X_\beta^*,\mathcal{B})$. By the definition it is not difficult to verify the following lemmas:

Lemma 3.1 *Under above conceptions we have*

1. *If for each $f \in X_\beta^*$ and $B \in \mathcal{B}$, $f(B)$ is a bounded subset of R^+, then $(X_\beta^*, \mathcal{B})$ is a topological cone.*

2. *If $\mathcal{B}$ still satisfies the stretching property: $\forall B \in \mathcal{B}$ and $r > 0$ we have $rB \in \mathcal{B}$, then $\{B^0 : B \in \mathcal{B}\}$ constitues a $\theta-$neighborhood basis of $(X_\beta^*, \mathcal{B})$. Where*

$$B^0 = \{f \in X_\beta^* : f(x) \leq 1, x \in B\}$$

is the polar set of B.

Let $\mathcal{B}_f$ and $\mathcal{B}_b$ denotes the family of all finite subsets and all bounded subsets of X respectively. Then $\mathcal{B}_f$ and $\mathcal{B}_b$ satisfy the conditions of lemma 3.1, hence $(X_\beta^*,\mathcal{B}_f)$ and $(X_\beta^*, \mathcal{B}_b)$ turn into two topological cones. We call the conical topology of $(X_\beta^*, \mathcal{B}_f)$ the UF-topology and the conical topology of $(X_\beta^*, \mathcal{B}_b)$ the $UB-$topology. Because $\mathcal{B}_f \subset \mathcal{B}_b$, the UF-topology is weaker than the UB-topology, and hence any UB-bounded subset of X_β^* (under the UB-topology) is certainly UF-bounded. Now we discuss the problem of when the UF-bounded subsets of X_β^* are also UB-bounded, i.e., the problem of when the uniform boundedness principle holds in topological cone X_β^*.

From the definitions it is easy to show the following lemmas.

Lemma 3.2 Let $B \subset X, H \subset X_\beta^*$. Then B^0 absorbs H if and only if H^0 absorbs B. Where

$$H^0 = \{x \in X : f(x) \leq 1, f \in H\}$$

is the polar set of H.

Lemma 3.3 *Let $\mathcal{B}$ be a directed family of subsets of a locally β-convex space X. If $\mathcal{B}$ have the streching property such that $(X_\beta^*, \mathcal{B})$ constitues a topological cone, then a set H is bounded in $(X_\beta^*, \mathcal{B})$ if and only if for every $B \in \mathcal{B}$, H^0 absorbs B.*

Theorem 3.1 *Let X be a locally β-convex space. If X is a β-barreled space (i.e., every β-barrel is a θ-neighborhood) or a Bair space, then the UF-boundedness and the UB-boundedness are equivalent in X_β^*.*

Proof We need merely to show that a UF-bounded set H is $UB-$bounded as well. From the continuity of the members of H, H^0 is a closed β-convex subset of X. By the UF-boundedness of H, we know from lemma 3.3 that H^0 has absorbing property. Thus $K = H^0 \bigcap (-H^0)$ is a β-barrel. If X is β-barreled, then K is naturally a θ-neighborhood. If X is a Bair space then from the absorbing property of K we have $X = \bigcup_{n=1}^{\infty} nK$. By the fact that K is closed and a Bair space is of itself second category there is $n_0 \in N$ such that $\text{int}(n_0K) \neq \Phi$, and hence $\text{int}K \neq \Phi$. Since K is circled, we have $\theta \in \text{int}K$, i.e., K is also a θ-neighborhood. For every bounded set $B \in \mathcal{B}_b$, B is absorbed by $K \subset H^0$. By lemma 3.2 we know that H is absorbed by B^0. Because $\{B^0 : B \in \mathcal{B}_b\}$ is a θ-neighborhood basis of $(X_\beta^*, \mathcal{B}_b)$, H is bounded in $(X_\beta^*, \mathcal{B}_f)$. □

Now we introduce the conception of subcomplete for a locally β-conves space X, and show that the subcompleteness is a necessary and sufficient condition for the equivalence of the UF-boundedness and the UB-boundedness in its conjugate cone X_β^*.

Definition 3.2 *Let X be a locally β-convex space, A an absolutely β-convex subset of X and $X_A = \text{lin}A$. A is called a β-disc if (X_A, P_{A_β}) is a complete β-normed space. Where*

$$P_{A_\beta}(x) := \inf\{t > 0 : x \in t^\beta A\}, x \in X_A$$

is the β-Minkowski functional generated by A. A locally β-convex space X is called subcomplete if each bounded subset of X is contained in some bounded β-disc.

Lemma 3.4[5] *Let T and T' be two vector topologies for a vector space X with T' being F-linked to T (i.e., there is a θ-neighborhood basis of (X, T') consisting of closed subsets of (X, T)). Then for each Cauchy net $\{x_\alpha\}$ in (X, T') we have $x_\alpha \xrightarrow{T'} x_0$ whenever $x_\alpha \xrightarrow{T} x_0$.*

Theorem 3.2 *Let X be a locally β-convex space. If X is subcomplete, then the UF-boundedness and the UB-boundedness are equivalent in X_β^*.*

Proof Let $H \subset X_\beta^*$ be UF-bounded. For every $B \in \mathcal{B}_b$, by the subcompleteness of X there is a bounded β-disc $A \subset X$ such that $B \subset A$. By the boundedness of A we know that the topology of X_A generated by the β-norm P_{A_β} is stronger than the induced topology $\tau|X_A$, where τ is the original vector topology of X. Let $H_{X_A} = \{f|_{X_A} : f \in H\}$ be the restriction of H on X_A, then from $H_{X_A} \subset (X_A, \tau|X_A)_\beta^*$ we obtain $H_{X_A} \subset (X_A, P_{X_A})_\beta^*$. Because H is UF-bounded in X_β^* we know that H_{X_A} is UF-bounded in $(X_A, \tau|X_A)_\beta^*$, then it is UF-bounded in $(X_A, P_{X_A})_\beta^*$. From the completeness we know that (X_A, P_{X_A}) is a Bair space. By theorem 3.1 we obtain that H_{X_A} is UB-bounded in $(X_A, P_{X_A})_\beta^*$ as well. Since $B(\subset A)$ is bounded in (X_A, P_{X_A}), B^0 absorbs H_{X_A} in $(X_A, P_{X_A})_\beta^*$, hence B^0 absorbs H in X_β^*. Thus H is UB-bounded in X_β^*. □

Theorem 3.3 *Let X be a locally β-convex Hausdorff space such that its every finite codimensional subspace has completmentary subspace. Then the UF-boundedness is equivalent to the UB-boundedness in X_β^* if and only if X is subcomplete.*

Proof By theorem 3.2 it is sufficient to show the necessity. Assume to the contrary, that X is not subcomplete. Since the closed absolutely β-convex hull of any bounded subset in a locally β-convex space is also bounded, there exists a bounded closed absolutely β-convex subset A of X such that (X_A, P_{A_β}) is not complete. Let $\{x_n\}_1^\infty$ be a Cauchy sequence in (X_A, P_{A_β}) that is not convergent. It is clear that $\{x_n\}_1^\infty$ is bounded in (X_A, P_{A_β}), and hence in X. Without loss of generality we may assume $\{x_n\}_1^\infty \subset A$. Let τ is the original topology of X, then the β-norm topology of (X_A, P_{A_β}) is F-linked to the induced topology $\tau|X_A$. By lemma 3.4, $\{x_n\}_1^\infty$ is not convergent in $(X_A, \tau|X_A)$ and hence in X. Thus $\{x_n\}_1^\infty$ can not be contained in any finite dimensional subspace of X. Assume without loss of generality that $P = \{x_n\}_1^\infty$ is linearly independent. Let Q be another linearly independent subset such that $P \bigcup Q$ constitues a Hamel basis of X. Since X is separated and every finite codimensional subspace of it has completmentary subspace, for each $n \in N$,

$$X_n = \mathrm{lin}[(P \bigcup Q) \backslash \{x_n\}]$$

is a closed subspace of X and $x_n \notin X_n$. Since $\{x_n\}$ is a compact singleton and X_n is a closed $\beta-$convex set and $x_n \notin X_n$, by the proof process of the second separation theorem we know that there is an open β-convex θ-neiborhood U of X such that

$$(X_n + U) \bigcap (\{x_n\} + U) = \Phi. \tag{6}$$

Let $f_n = P_{(X_n+U)_\beta}$, then from $\theta \in int(X_n + U)$ and lemma 2.1 we have $f_n \in X_\beta^*$. By the definition and (6) we have

$$f_n(x) = 0, \forall x \in X_n; f_n(x_n) > 0, \forall n \in N. \tag{7}$$

Let $g_n(x) = \frac{1}{f_n(x_n)} f_n(x)$, then $g_n \in X_\beta^*$ and

$$g_n(x) = 0, \forall x \in X_n; g_n(x_n) = 1, \forall n \in N. \tag{8}$$

Let $H = \{ng_n\}_1^\infty \subset X_\beta^*$. Then for every element

$$x = \sum_{i=1}^{m} \xi_i x_i + y \in X, y \in \mathrm{lin}Q,$$

when $n > m$ we have $x \in X_n$, $g_n(x) = 0$. So for every finite subset B of X, there exists $M < +\infty$ such that

$$\sup_{x \in B} \sup_{n \in N} \{ng_n(x)\} < M. \tag{9}$$

Thus $\frac{1}{M} H \subset B^0$, i.e., H is UF-bounded in X_β^*. But for the bounded subset $P = \{x_n\}_1^\infty$ of X we have

$$\sup_{x \in P} \sup_{n \in N} \{ng_n(x)\} = +\infty. \tag{10}$$

This is to say that H can be absorbed by P^0, and H is not UB-bounded in X_β^*. This is contrary to the fact that the UF-boundedness is equivalent to the UB-boundedness in X_β^*. □

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局部 β- 凸空间中的第二分离性定理及其共轭锥上的有界性定理

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摘 要: 第一部分给出局部 β- 凸空间中的第二分离性定理和 Minkowski 定理及 Krein-Milman 定理等；第二部分得到其共轭锥上 UF- 有界与 UB- 有界等价的充要条件为原空间是次完备的.