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Equilibrium Small Circuit Double Covers of Near-Triangulations *

LIU Tong-yin, *LIU Yan-pei*
(Dept. of Math., Northern Jiaotong University, Beijing 100044, China)

Abstract: A near-triangulation is such a connected planar graph whose inner faces are all triangles but the outer face may be not. Let G be a near-triangulation of order n and $\mathcal{C}$ be an SCDC (small circuit double cover)[2] of G. Let

$$\delta(\mathcal{C}_0) = \min\{\max_{c_j \in \mathcal{C}}\{l(c_j)\} - \min_{c_j \in \mathcal{C}}\{l(c_j)\} \mid \mathcal{C} \text{ is an SCDC of } G\}.$$

Then, $\mathcal{C}_0$ is said to be an equilibrium SCDC of G. In this paper, we show that if G is an outer planar graph, $\delta(\mathcal{C}_0) \le 2$, otherwise $\delta(\mathcal{C}_0) \le 4$.

Key words: small circuit double cover; near-triangulation.

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1. Introduction

All graphs considered here are finite, undirected and simple (without loops or multiple edges). For a polyhedron $\sum$, the graph $G(\sum) = (V(\sum), E(\sum))$, where $V(\sum)$ and $E(\sum)$ are the sets of vertices and edges of $\sum$ respectively, is called the underlying graph of $\sum$ and $\sum$, an underlain polyhedron of $G(\sum)$. For a graph $G = (V, E)$, if there is a polyhedron $\sum_{\sim El} S \in \mathbf{S}$ such that G is the underlying graph of $\sum$, then G is said to be embeddable on the surface S. The polyhedron is called an *embedding* of G. If a graph has an embedding in the plane, then it is said to be *planar*. The boundary of a face f is denoted by ∂f. Except the outer face of planar graph G, the other faces are called inner faces. If a vertex v of G does not belong to the outer face of G, then v is said to be an interior vertex of G. Let c be a circuit of G. If the length of c is $l(c)$, then the circuit is said to be an l-circuit. Similarly, if the degree of a vertex is d, it is called a d-vertex. Terminologies and notations not explained here can be seen in [1].

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Let $\mathcal{C}$ be a collection of circuits of a graph G. If each edge of G belongs to exactly two members of $\mathcal{C}$, then $\mathcal{C}$ is said to be a circuit double cover (CDC) of graph G. For a planar graph G of order n, if G admits a CDC such that $|\mathcal{C}| \leq n-1$, then $\mathcal{C}$ is said to be a small circuit double cover (SCDC)[2] of G.

A triangulation is a simple planar graph in which every face is a triangle. A near-triangulation is such a connected planar graph whose inner faces are all triangles but the outer face may be not. If G is a planar graph and G has cut-vertices, we are allowed to consider the blocks of G for CDC or SCDC. So all near-triangulations considered here are 2-connected. Clearly, a triangulation is also a near-triangulation. If G is a planar graph and it has no interior vertex, it is called an outer planar graph.

Let $G=(V,E)$ be a graph. A minor[3] is a graph obtained from G by successively deleting and/or contracting edges of G while any multiple edge that might occur is replaced by a single edge and a loop by its end. A generator is a graph obtained from G by successively deleting vertices with their incident edges. Clearly, for a graph, its generator is also a minor.

Let $G=(V,E)$ be a 2-connected near-triangulation of order n and f be the outer face of G. If G is an outer planar graph and G contains a triangle T in which $\forall e \in E(T)$, $e \notin E(\partial f)$, then T is said to be an interior triangle of G. Let $G=(V,E)$ be a near-triangulation with at least one interior vertex u and $|V(G)|=n$. Then, $G[u \bigcup N(u)]$ is a wheel. In fact, if G has only one interior vertex, then G is a wheel by itself. Otherwise there is at least one vertex v which is not adjacent to u by the planarity of G.

Let v_0 be a vertex such that

$$d(u,\ v_0)=\max\{d(u,\ v) \mid d(u,\ v) \geq 2, v \in \partial f\},$$

where $d(u,\ v)$ is the distance between u and v. Then, we delete the vertex v_0 with its incident edges. Repeating the procedure, a wheel can be recursively obtained. Conversely, any near-triangulation can be obtained by successively adding a new vertex to the outer face with its incident edges. So $G[u \bigcup N(u)]$ is said to be a wheel generator of G, denoted by W^u.

In previous papers[4, 5], we sketched the proofs of the following results.

Theorem 1.1[4] *Every near-triangulation admits an* SCDC.

Theorem 1.2[5] *Let G be an outer planar graph of order n and $\mathcal{C}$ is an* SCDC *of G. If G is a near-triangulation, then $|\mathcal{C}| \leq n-2$ if and only if G contains an interior triangle.*

We know that simple planar graphs on n vertices may have as many as $3n-6$ edges. Suppose G admits an SCDC $\mathcal{C}$. Since every edge of G is contained in precisely two circuits of $\mathcal{C}$, we have $\sum_{i=1}^{m} l(c_i) \leq 6n-12$, where $m=|\mathcal{C}|$ and $c_i \in \mathcal{C}$. Suppose $\bar{l}(c_i)$ is the average length of circuits of $\mathcal{C}$, then we have $\bar{l}(c) \cdot m \leq \bar{l}(c) \cdot (n-1) \leq 6n-12$ and

$$\bar{l}(c) \leq 6-\frac{6}{n-1}.$$

But what is the length of the longest circuit in $\mathcal{C}$? It seems that the problem has not been treated in literatures. Here we consider the compactness of an SCDC $\mathcal{C}$ of G. Let

$$\delta(\mathcal{C}_0)=\min\{\max_{c_j \in \mathcal{C}}\{l(c_j)\}-\min_{c_j \in \mathcal{C}}\{l(c_j)\} \mid \mathcal{C} \text{ is an SCDC of } G\}.$$

Then, $\mathcal{C}$ is said to be an equilibrium small circuit double cover. For a near-triangulation G of order n, if G is an outer planar graph, we will prove that $\delta(\mathcal{C}_0) = 2$; if G has at least one interior vertex, then $\delta(\mathcal{C}_0) \leq 4$.

2. Main results

Before obtaining the main results of the section, we provide some useful definitions and lemmas.

Let G be a planar near-triangulation of order n ($n \geq 4$) and $\mathcal{C}$ be an SCDC of G. Let f be the outer face of G and $\partial f = v_1 v_2 \cdots v_k v_1$. If there exists circuits $c_1, c_2, \cdots, c_k$ in $\mathcal{C}$, such that $c_i \neq c_j$ for $i \neq j$ and $v_i v_{i+1} \in c_i$ $(i, j = 1, 2, \cdots, k, v_{k+1} = v_1)$, then $\mathcal{C}$ is said to be a fair small circuit double cover, denoted by FSCDC. The circuit c_i is called an associated circuit of $v_i v_{i+1}$ of f. The set of all associated circuits is said to be the associated set of f, denoted by $\mathcal{C}_f$. Let $\mathcal{C}$ be an FSCDC of G. From the definition of FSCDC, we know that for $c_i, c_j \in \mathcal{C}(i \neq j, i, j = 1, 2, \cdots, m, m \leq n-1)$, $|E(c_i) \cap E(c_j) \cap E(f)| \leq 2$.

If G is an outer planar graph, then it is a Hamiltonian graph, $|\partial f| = n$ and $|\mathcal{C}| \leq n-1$. So G does not admit an FSCDC. If G contains at least one interior vertex, then we have the following lemma.

Lemma 2.1[4] *Every near-triangulation with at least one interior vertex admits an* FSCDC.

Lemma 2.2[5] *Let G be an outer planar near-triangulation of order n with exactly two 2-vertices. Then, for any* SCDC *$\mathcal{C}$ of G, there are at least two 3-circuits in $\mathcal{C}$ and $|\mathcal{C}| = n-1$.*

In what follows, let $L(\mathcal{C}) = \max\{l(c_i) | \forall c_i \in \mathcal{C}\}$ and $l(\mathcal{C}) = \min\{l(c_i) | \forall c_i \in \mathcal{C}\}$, where $\mathcal{C}$ is an SCDC of graph G.

Lemma 2.3 *Let G be an outer planar near-triangulation on n vertices with only two 2-vertices. Then, G admits an* SCDC *$\mathcal{C}$ such that $L(\mathcal{C}) \leq 4$.*

Proof Let f be the outer face of G. Suppose u, v are the two 2-degree vertices, $N(u) = \{u_1, u_2\}$ and $N(v) = \{v_1, v_2\}$. Since G has $n-1$ faces and each face's boundary is a circuit, let $\mathcal{C}_1$ be the collection of all the circuits. Then $\mathcal{C}_1$ is composed of $n-2$ 3-circuits and ∂f. We denote the collection of the $n-2$ 3-circuits by $\mathcal{C}_0 = \{c_i | i = 1, 2, \cdots n-2\}$. It is easy to see that each edge of G is contained in precisely two members of $\mathcal{C}_1$ and $|\mathcal{C}_1| = n-1$, hence, $\mathcal{C}_1$ is an SCDC of G. As any two adjacent circuits of $\mathcal{C}_0$ have a common edge, let $c_i{}' = (\partial c_i \bigtriangleup \partial c_{i+1})$ $(i = 1, 2, \cdots, n-3, c_i \in \mathcal{C}_0)$. Let $\mathcal{C}_2 = \{c_i{}' \mid i = 1, 2, ..., n-3\}$. Then, almost every edge of G belongs to two circuits of $\mathcal{C}_2$ with the exception of the edges $u_1 u_2$, $v_1 v_2$, uu_i and $vv_j (i, j = 1, 2)$ which are contained in only one member of $\mathcal{C}_2$ respectively. Adding two 3-circuits uu_1u_2u,vv_1v_2v to $\mathcal{C}_2$, we get the collection $\mathcal{C}$ as following

$$\mathcal{C} = \mathcal{C}_2 \bigcup \{uu_1u_2u, vv_1v_2v\}.$$

Then, $\mathcal{C}$ is a CDC of G and $|\mathcal{C}| = n-1$. It can be seen that $L(\mathcal{C}) = 4$ and $l(\mathcal{C}) = 3$. Hence the lemma holds. □

Let $\mathcal{C}$ be an SCDC of G and f be the outer face of G. For an edge e of ∂f, c_1 and c_2 are the two circuits that cover the edge e. Then, the shorter one is said to be the feasible circuit of e.

Lemma 2.4 *Let G be an outer planar near-triangulation of order n ($n \geq 5$) with at least three 2-vertices. Then G admits an SCDC $\mathcal{C}$ such that $L(\mathcal{C}) = 5$.*

Proof Let f be the outer face of G. Suppose G has k ($k \geq 3$) 2-vertices, clearly, the k 2-vertices belong to ∂f because otherwise G would have a multi-edge. We delete one 2-vertex v_0 of ∂f, along with its incident edges, denoted the resulting graph by G_1 whose outer face is denoted by f_1. If G_1 has exactly two 2-vertices, then we let $G^* = G_1$ and $f^* = f_1$. Otherwise we delete one 2-vertex v_1 of ∂f_1, along with its incident edges, denoted the resulting graph by G_2 whose outer face is denoted by f_2. Since G is a near-triangulation without interior vertex and it contains at least two 2-vertices that only belong to the boundary of the outer face of G, we can obtain G^*, which is a subgraph of G with only two 2-vertices, after repeating the above procedure in finite steps. Without loss of generality, the deleted 2-vertices are denoted in the order of $v_0, v_1, \cdots, v_{m-1}$ and the resulting graphs $G_1, G_2, \cdots, G_m$, respectively. Let n^* be the order of G^*. By the reason argument used in the proof of Lemma 2.2, G^* admits an SCDC $\mathcal{C}^*$ such that $L(\mathcal{C}^*) = 4$, denoted the collection of 4-circuits of $\mathcal{C}^*$ by $\mathcal{C}_1$. Then $|\mathcal{C}_1| = n^* - 3$.

Considering the reverse procedure of obtaining G^* from G, we add the deleted m 2-vertices $v_{m-1}, v_{m-2}, \cdots, v_0$ along with their incident edges to $G_m, G_{m-1}, \cdots, G_1$ to obtain the original graph G. Then the boundaries of new added triangles are 3-circuits, denoted by $c_m, c_{m-1}, \cdots, c_1$ respectively and $\mathcal{C}_2$, the set of all the 3-circuits. Hence, $|\mathcal{C}_2| = m$. Meanwhile, we modify $\mathcal{C}^*$ to obtain $\mathcal{C}$ as follows. Let $e_i = \partial c_i \bigcap E(G_i)$ $(i = 1, 2, \cdots, m)$. Then, we take

$$\mathcal{C}_m = (\mathcal{C}^* \backslash \{c_m^*\}) \bigcup \{\partial c_m^* \triangle \partial c_m,\ c_m\},$$

where c_m^* is the feasible circuit of e_m in $\mathcal{C}^*$. It can be checked that $\mathcal{C}_m$ is an SCDC of G_m. Since the length of the feasible circuit is at most 4, $L(\mathcal{C}_m) = 5$ and $l(\mathcal{C}_m) = 3$. Then we modify $\mathcal{C}_{i+1}$ to obtain $\mathcal{C}_i$ in the same way $(i = m-1, m-2, \cdots, 2, 1)$. At last, let $\mathcal{C}_1$ be $\mathcal{C}$. It can be checked that $\mathcal{C}$ is an SCDC of G. Notice that the length of the modified feasible circuit in each procedure of obtaining G is at most 4 and there exists at least one 3-circuits of the SCDC of the resulting graph, $L(\mathcal{C}) = 5$ and $l(\mathcal{C}) = 3$. Hence, the lemma holds. □

Corallary 2.1 *Let G be an outer planar 2-connected near-triangulation of order n with k ($k \geq 2$) 2-vertices. Then, G admits an SCDC $\mathcal{C}$, such that $|\mathcal{C}| = n - 1$ and $|\mathcal{C}_1| = k$, where $\mathcal{C}_1$ is the collection of 3-circuits in $\mathcal{C}$.*

Let G be an outer planar graph on n vertices and $\mathcal{C}$ be an SCDC of G. If G is a near-triangulation, then $|E(G)| = 2n - 3$. Moreover, $\sum_{i=1}^{m} l(c_i) = 4n - 6$, where $m = |\mathcal{C}|$ and $m \leq n - 1$, $c_i \in \mathcal{C}$. So we have $\bar{l}(c) \cdot m \leq \bar{l}(c) \cdot (n-1) \leq 4n - 6$ and

$$\bar{l}(c) \leq 4 - \frac{2}{n-1}. \tag{1}$$

From Lemma 2.4, we know that G admits an SCDC $\mathcal{C}$, such that 5 is the upbound for $L(\mathcal{C})$.

Theorem 2.1 *Let G be an outer planar graph of order n and n_1 be the number of 2-vertices of G. If G is a near-triangulation, then G admits an equilibrium small circuit double cover $\mathcal{C}_0$, such that*

(a) $\delta(\mathcal{C}_0) = 0$, *if* $n = 3$;

(b) $\delta(\mathcal{C}_0) = 1$, *if* $n_1 = 2$ *and* $n \geq 4$, *or* $n_1 = 3$ *and* $n = 6$;

(c) $\delta(\mathcal{C}_0) \leq 2$, *if* $n_1 \geq 3$ *and* $n \geq 7$.

Proof Let f be the outer face of G. Since G is an outer planar near-triangulation, it contains at least two 2-vertices. If $n = 3$, *i.e.*, G is a triangle, then $L(\mathcal{C}_0) = l(\mathcal{C}_0) = 3$. Hence (a) follows.

If G contains only two 2-vertices and G is not a triangle, then $|\partial f| \geq 4$. Let $\mathcal{C}$ be an SCDC of G. By Lemma 2.2, $|\mathcal{C}| = n - 1$ and there are at least two 3-circuits in $\mathcal{C}$. Therefore, $l(\mathcal{C}) = 3$. For any SCDC $\mathcal{C}$ of G, the number of 3-circuits of SCDC $\mathcal{C}'$ obtained from Lemma 2.3 is the smallest, that is 2, and all other circuits are 4-circuits. Then, from (1) and the definition of equilibrium small circuit double cover, $\delta(\mathcal{C}_0) = 1$.

If $n_1 = 3$ and $n = 6$, then from Theorem 1.2, G admits an SCDC $\mathcal{C}$ such that $|\mathcal{C}| \leq n - 2 = 4$. In fact, if $|\mathcal{C}| < 4$, the length of each circuit of $\mathcal{C}$ is at least 6 since $|E(G)| = 9$. Since $|V(G)| = 6$, only one 6-circuit can be contained in G, i.e., ∂f. Then, $\mathcal{C}$ is not an SCDC of G. So if $|\mathcal{C}| \leq 4$, then $|\mathcal{C}| = 4$. Let T_0 be the interior triangle and T_1, T_2 and T_3 be the three triangles around T_0 clockwisely. Let

$$c_i = \partial T_0 \triangle \partial T_i \triangle \partial T_{i+1} \ (i = 1, 2),$$

$$c_3 = \partial T_0 \triangle \partial T_1,$$

$$c_4 = \partial T_0 \triangle \partial T_3.$$

Let

$$\mathcal{C} = \{c_i \mid i = 1, 2, 3, 4\}.$$

where $l(c_i) = 5$ $(i = 1, 2)$ and $l(c_i) = 4$ $(i = 3, 4)$. It can be seen that each edge of G is contained in exactly two circuits of $\mathcal{C}$. So G admits an SCDC $\mathcal{C}$ such that $|\mathcal{C}| = 4$ and $L(\mathcal{C}) = 5$, $l(\mathcal{C}) = 4$. Since $2|E(G)| = 18$, there is no SCDC of G such that it contains five circuits having equal length or four circuits with same length. So from the definition of equilibrium small circuit double cover of G, $\delta(\mathcal{C}_0) = 1$. These imply (b).

If $n_1 \geq 3$ and $n \geq 7$, by Lemma 2.4, G admits an SCDC $\mathcal{C}^*$ such that $|\mathcal{C}^*| \leq n - 1$ and $L(\mathcal{C}^*) = 5$, $l(\mathcal{C}^*) = 3$. By the definition of equilibrium small circuit double cover, $\delta(\mathcal{C}_0) \leq 2$.

The proof is completed. □

Lemma 2.5 *Every wheel admits an equilibrium small circuit double cover $\mathcal{C}_0$, such that* $\delta(\mathcal{C}_0) = 0$.

Proof Let G be a wheel of order n. Let $f_1, f_2, \cdots, f_{n-1}$ be the $n - 1$ inner faces of G in clockwise. Take

$$\mathcal{C} = \{\partial f_i \triangle \partial f_{i+1}) \mid i = 1, 2, \cdots, n - 1\}$$

where the addition in the suffixes is to be modulo $n-1$. It is easy to see that $\mathcal{C}$ is an SCDC such that $|\mathcal{C}| = n-1$ and $L(\mathcal{C}) = l(\mathcal{C}) = 4$. Meanwhile, it can be seen that $\mathcal{C}$ is also an FSCDC from Lemma 2.1. Hence, we proved the lemma. □

Lemma 2.6 *Let G be a planar near-triangulation of order n with at least one interior vertex u. Let W^u be the wheel generator of G and $|V(W^u)| \geq 5$. If there is at least one vertex v such that $d(u,v) \geq 2$, then G admits an* SCDC *$\mathcal{C}$ such that $L(\mathcal{C}) \leq 7$, $l(\mathcal{C}) = 3$ except for $n = 6$ with $\rho(v) = 2$, $L(\mathcal{C}) = l(\mathcal{C}) = 5$.*

Proof In the reverse procedure of obtaining W^u, we firstly add a d_1-vertex v with its incident d_1 edges to W^u. Denote the resulting graph by G_1. By Lemma 2.5, W^u admits an SCDC $\mathcal{C}^*$, such that $|\mathcal{C}^*| = |V(W^u)| - 1$ and $L(\mathcal{C}^*) = l(\mathcal{C}^*) = 4$. For G_1, there are two cases.

Case 1. $d_1 > 2$.

Let the neighbors of new added d_1-vertex v be $v_1, v_2, \cdots, v_{d_1}$ in sequence. Since $\mathcal{C}$ is also an FSCDC, corresponding to each edge $v_i v_{i+1}$ $(i = 1, 2, \cdots, d_1 - 1)$ of W^u, the associated circuit of $\mathcal{C}^*$ are denoted by $c_1, c_2, \cdots, c_{d_1 - 1}$, which contain the edge uv_i $(i = 1, 2, \cdots, d_1 - 1)$ respectively. We modify $\mathcal{C}^*$ to obtain the collection $\mathcal{C}_1$ of G_1 as follows: we replace the segment $v_1 v_2$ of c_1 by the path $v_1 v v_2$ and denote the resulting circuit by c_1^1. Then, we replace the segment $v_i v_{i+1}$ of c_i by the path $v_i v_{i-1} v v_{i+1}$ and denote the resulting circuits by $c_i^1 (i = 2, 3, \cdots, d_1 - 1)$. Finally, we add a 3-circuit $v_{d_1} v_{d_1 - 1} v v_{d_1}$ to $\mathcal{C}^*$, denoted by c_{d_1}. Then, we get $\mathcal{C}_1$

$$\mathcal{C}_1 = (\mathcal{C}^* \backslash \{c_i | i = 1, 2, \cdots, d_1 - 1\}) \bigcup \{c_i^1 | i = 1, 2, \cdots, d_1 - 1\} \bigcup \{c_{d_1}\}.$$

It is easy to see that the length of the longest circuit of $\mathcal{C}_1$ is 6, $l(c_1^1) = 5$, $l(c_{d_1}) = 3$ and the length of other circuits of $\mathcal{C}_1$ is 4.

Case 2. $d_1 = 2$.

We follow the notations of Case 1 and let $N(v) = \{v_1, v_2\}$. Corresponding to the edge $v_1 v_2$ of W^u, the associated circuit of $\mathcal{C}^*$ is c_1. We modify $\mathcal{C}^*$ to obtain the collection $\mathcal{C}_1$ of G_1 as follows: replacing the segment $v_1 v_2$ of c_1 by the path $v_1 v v_2$, denote the resulting circuit by c_1^1, and add a new 3-circuit $v_1 v_2 v v_1$ to $\mathcal{C}^*$, denoted by c_d. Then, we get $\mathcal{C}_1$ as follows

$$\mathcal{C}_1 = (\mathcal{C}^* \backslash \{c_1\}) \bigcup \{c_1^1, c_d\}.$$

It is easy to see that $L(\mathcal{C}_1) = 5 = l(c_1^1)$ and $l(c_d) = 3$.

For $d_1 = 2$ or $d_1 > 2$, every edge of G_1 is contained in precisely two circuits of $\mathcal{C}_1$ and $L(\mathcal{C}_1) \leq 6$. So $\mathcal{C}_1$ is an SCDC of G_1 and $L(\mathcal{C}_1) \leq 6$. From the definition of FSCDC and the procedure of obtaining $\mathcal{C}_1$, $\mathcal{C}_1$ is also an FSCDC of G. Let the outer face of G_1 be f_1. Next we add a d_2-vertex, along with d_2 edges to G_1, denote the resulting graph by G_2, which is also a near-triangulation. Then, we modify the feasible circuit(s) of ∂f_1 of $\mathcal{C}_1$, in which the ends of the feasible edges is adjacent to the new added d_2-vertex, to obtain $\mathcal{C}_2$ by applying the previous method. Clearly, $\mathcal{C}_2$ is an FSCDC of G_2. From the procedure of obtaining $\mathcal{C}_1$, it can be seen that, for the edges in the outer face boundary of G_1, the lengths of feasible circuits of $\mathcal{C}_1$ are 3, 4 or 5. Since in the procedure of modifying $\mathcal{C}_1$ to $\mathcal{C}_2$, the length of each circuit of $\mathcal{C}_1$ increase at most 2, and $\mathcal{C}_2$ must contain a 3-circuit, the longest circuit of $\mathcal{C}_2$ is at most 7 and $l(\mathcal{C}_2) = 3$. Meanwhile, the length of feasible circuit of each edge belonging to the outer face boundary of G_2 can also be 3, 4 or 5 from the procedure of obtaining $\mathcal{C}_2$.

Repeating the above procedure, we can recursively get $\mathcal{C}$. It may be verified that $\mathcal{C}$ is an SCDC of G and $L(\mathcal{C}) \leq 7$ and $l(\mathcal{C}) = 3$.

If $n = 6$ and $\rho(v) = 2$, we can obtain an SCDC $\mathcal{C}$ by a similar metholu, such that $\mathcal{C}$ is composed of four 5-circuits. □

By Lemma 2.5 and Lemma 2.6, the following result can be directly proved.

Theorem 2.2 *Let G be a planar near-triangulation of order n with at least one interior vertex u. Let W^u be the generating wheel of G and $|V(W^u)| \geq 5$. Then, G admits an equilibrium small circuit double cover $\mathcal{C}_0$, such that, if $G = W^u$ or $n_1 = n_2 = 1$ and $n = 6$, $\delta(\mathcal{C}_0) = 0$; otherwise $\delta(\mathcal{C}_0) \leq 4$, where n_1 and n_2 are the numbers of 2-vertices and interior vertices of G respectively.*

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近三角剖分图的均衡二重少圈覆盖

刘同印， 刘彦佩

(北方交通大学数学系，北京 100044)

摘 要: 近三角剖分图是一连通平面图，其内面均为三角形而其外面可能不是. 令 G 为一具有 n 个节点的近三角剖分图， $\mathcal{C}$ 为 G 的一个小圈二重覆盖 (SCDC)[2]. 令

$$\delta(\mathcal{C}_0) = \min\{\max_{c_j \in \mathcal{C}}\{l(c_j)\} - \min_{c_j \in \mathcal{C}}\{l(c_j)\} \mid \mathcal{C} \text{ 为 G 的一个 SCDC}\},$$

则 $\mathcal{C}_0$ 称为 G 的均衡小圈二重覆盖. 本文将证明：若 G 为外平面图，则 $\delta(\mathcal{C}_0) \leq 2$; 否则 $\delta(\mathcal{C}_0) \leq 4$.