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The Nonlocal Singularly Perturbed Problems for Nonlinear Hyperbolic Differential Equation *

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Abstract: A class of nonlocal singularly perturbed problems for the hyperbolic differential equation are considered. Under suitable conditions, we discuss the asymptotic behavior of solution for the initial boundary value problems.

Key words: nonlocal problem; singular perturbation; nonlinear hyperbolic differential equation.

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1. Introduction

We consider the nonlocal singularly perturbed problem as follows

$$\frac{\partial^2 u}{\partial t^2} - \varepsilon^3 L_1 u - \varepsilon L_2 u = f(t,x,u,Tu,\varepsilon),(t,x) \in (0,T_0] \times \Omega, \tag{1}$$

$$u = g(t,x,\varepsilon), x \in \partial\Omega, \tag{2}$$

$$u\mid_{t=0} = h_1(x,\varepsilon), \tag{3}$$

$$\frac{\partial u}{\partial t}\mid_{t=0} = h_2(x,\varepsilon), \tag{4}$$

where ε is a positive small parameter and

$$L_1 = \sum_{i,j=1}^{n} \alpha_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}, \sum_{i,j=1}^{n} \alpha_{ij}(x)\xi_i\xi_j \ge \lambda_i \sum_{i=1}^{n} \xi_i^2, \quad \forall \xi_i \in R, \lambda_i > 0,$$

$$L_2 = \sum_{i=1}^{n} \beta_i(x)\frac{\partial}{\partial x_i}, \sum_{i=1}^{n} \beta_i(x) n_i \ge \delta > 0,$$

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$$Tu \equiv \phi(x) + \int_\Omega K(x,y)udy, K(x,y) \geq 0, x, y \in \Omega,$$

$x \equiv (x_1, x_2, \cdots, x_n) \in \Omega, \Omega$ denotes a bounded region in $R^n, \partial\Omega$ signifies a boundary of Ω for class $C^{1+\alpha}(\alpha \in (0,1)$ is Hölder exponent),n_i is a direction coefficient of the inner normal on $\partial\Omega$, T_0 is a positive constant, L_1 is a uniformly elliptic operator, L_2 is a first order differential operator,T is a integral operator, $K(x,y)$ is continuous function.

Problem (1) – (4) is a nonlinear initial boundary value problem. This paper involves a class of nonlocal singularly perturbed problems, constructs the asymptotic expansion of solution, and discusses its asymptotic behavior.

We need the following hypotheses:

[H_1] The first order partial derivations of $\alpha_{ij}, \beta_i, \phi$ with regard to x are Hölder continuous, and g and h_i with regard to x, t are Hölder continuous, with regard to ε are sufficiently smooth functions in correspondence ranges.

[H_2] $f(t,x,u,v,\varepsilon)$ is a sufficiently smooth function with regard to variables in correspondence ranges and

$$f_u(t,x,u,Tu,\varepsilon) \leq -c_1 \leq 0, f_{Tu}(t,x,u,Tu,\varepsilon) \leq -c_2 \leq 0,$$

where c_1 and c_2 are constants and $(c_1 + c_2) > 0$.

2. Construction formal asymptotic expansion

We now construct a formal asymptotic expansion for the solution of the problem (1)-(4).

The reduced problem for the original problem is

$$\frac{d^2u}{dt^2} = f(t,x,u,Tu,0), (t,x) \in (0,T_0] \times \Omega, \tag{5}$$

$$u\,|_{t=0} = h_1(x,0), x \in \Omega, \tag{6}$$

$$\frac{du}{dt}\,|_{t=0} = h_2(x,0), x \in \Omega. \tag{7}$$

From the hypotheses, there exists a unique solution $U_0(t,x)$ for the problem (5)–(7).

Let formal expansions of the outer solution $U(t,x,\varepsilon)$ for the original problem (1)–(4) be

$$U(t,x,\varepsilon) \sim \sum_{i=0}^{\infty} U_i(t,x)\varepsilon^i. \tag{8}$$

Substituting (8) into (1),(3) and (4), developing f and $h_i, i = 1,2$ in ε, equating coefficients of like powers of ε respectively, for $i = 1,2,\cdots$ we obtain

$$\frac{d^2U_i}{dt^2} = f_u(t,x,U_0,TU_0,0)U_i + f_{Tu}(t,x,U_0,TU_0,0)TU_i + L_1U_{i-3} + L_2U_{i-1} + F_i, \tag{9}$$

$$U_i\,|_{t=0} = h_{1i}, \tag{10}$$

$$\frac{dU_i}{dt}\,|_{t=0} = h_{2i}, \tag{11}$$

where

$$h_{ji} = \frac{1}{i!}[\frac{\partial^i h_j}{\partial \varepsilon^i}]_{\varepsilon=0}, j = 1, 2, i = 1, 2, \cdots,$$

and F_i are determinded functions of $U_k, k \le i-1$. Here and in what follows the values of terms for the negative subscript are zero. From the above linear problems(9)–(11), we can solve $U_i(t,x)$ successively. Then we obtain the outer solution $U(t,x,\varepsilon)$ for the original problem. But it may not satisfy the boundary conditions (2), so that we need to construct the boundary layer function V.

Set up a local coordinate system (ρ, φ) as [3], where $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_{n-1})$. In the neighborhood of $\partial\Omega : 0 \le \rho \le \rho_0$,

$$L_1 = a_{nn}\frac{\partial^2}{\partial \rho^2} + \sum_{i-1}^{n-1} a_{ni}\frac{\partial^2}{\partial\rho\partial\varphi_i} + \sum_{i-1}^{n-1} a_{ij}\frac{\partial^2}{\partial\phi_i\partial\varphi_j} + a_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} a_i\frac{\partial}{\partial\varphi_i}, \tag{12}$$

$$L_2 = b_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} b_i\frac{\partial}{\partial\varphi_i}, \tag{13}$$

where

$$a_{nn} = \sum_{i=1}^{n}(\frac{\partial\rho}{\partial x_i})^2, a_{ni} = 2\sum_{k=1}^{n}\frac{\partial\rho}{\partial x_k}\frac{\partial\varphi_i}{\partial x_k}, a_{ij} = \sum_{k=1}^{n}\frac{\partial\varphi_j}{\partial x_k}\frac{\partial\varphi_i}{\partial x_k},$$

$$a_n = \sum_{j=1}^{n}\frac{\partial^2\rho}{\partial x_j^2}, a_i = \sum_{j=1}^{n}\frac{\partial^2\varphi_i}{\partial x_j^2}, b_n = \sum_{j=1}^{n}\frac{\partial\rho}{\partial x_j}, b_i = \sum_{j=1}^{n}\frac{\partial\varphi_i}{\partial x_j}.$$

We lead to the variables of multiple scales[1] on $0 \le \rho \le \rho_0$:

$$\tau = \frac{h(\rho,\varphi)}{\varepsilon^2}, \bar\rho = \rho, \varphi = \varphi,$$

where $h(\rho,\varphi)$ is a function to be determined from (29). For convenience, we still substitute ρ for $\bar\rho$ below. From (12) and (13) we have

$$L_1 = \frac{1}{\varepsilon^4}K_{10} + \frac{1}{\varepsilon}K_{11} + K_{12}, L_2 = \frac{1}{\varepsilon^2}K_{20} + K_{21}, \tag{14}$$

where

$$K_{10} = a_{nn}h_\rho^2\frac{\partial^2}{\partial\tau^2},$$

$$K_{11} = 2a_{nn}h_\rho\frac{\partial^2}{\partial\tau\partial\rho} + \sum_{i=1}^{n-1} a_{ni}h_\rho\frac{\partial^2}{\partial\tau\partial\varphi_i} + (a_{nn}h_\rho + a_nh_\rho)\frac{\partial}{\partial\tau},$$

$$K_{12} = a_{nn}\frac{\partial^2}{\partial\rho^2} + \sum_{i=1}^{n-1} a_{ni}\frac{\partial^2}{\partial\tau\partial\varphi_i} + \sum_{i=1}^{n-1} a_{ij}\frac{\partial^2}{\partial\varphi_i\partial\varphi_j} + a_n\frac{\partial}{\partial\rho} + \sum_{i=1}^{n-1} a_i\frac{\partial}{\partial\varphi_i},$$

$$K_{20} = b_nh_\rho\frac{\partial}{\partial\tau},$$

$$K_{21} = b_n \frac{\partial}{\partial \rho} + \sum_{i=1}^{n-1} b_i \frac{\partial}{\partial \varphi_i}.$$

Let the solution u of original problem(1)-(4) be

$$u = U(t, x, \varepsilon) + V(t, \tau, \rho, \varphi, \varepsilon). \tag{15}$$

Substituting (15) into (1),(2), we have

$$V_{tt} - \varepsilon^3 L_1 V - \varepsilon L_2 V = f(x, U + V, T(U + V), \varepsilon) - f(x, U, TU, \varepsilon), \tag{16}$$

$$V\mid_{\rho=0} = g(t, x, T(U + V), \varepsilon) - U(t, x, \varepsilon), \quad x \equiv (\rho, \varphi) = (0, \varphi) \in \partial\Omega, \tag{17}$$

Let

$$V \sim \sum_{i=0}^{\infty} v_i(t, \tau, \rho, \varphi)\varepsilon^i. \tag{18}$$

Substituting (8),(18) and (15) into (16)-(17), expanding nonlinear terms in ε, and equating the coefficients of like powers of ε, we obtain

$$(K_{10} + K_{20})v_0 = 0, \tag{19}$$

and

$$(K_{10} + K_{20})v_1 = (v_0)_{tt} - G_0, \tag{20}$$

$$(K_{10} + K_{20})v_2 = (v_1)_{tt} - G_1, \tag{21}$$

$$(K_{10} + K_{20})v_i = (v_{i-1})_{tt} - (K_{11} + K_{21})v_{i-1} - K_{12}v_{i-3} - G_{i-1}, i = 3, 4, \cdots, \tag{22}$$

$$v_0\mid_{\rho=0} = g(t, 0, \varphi, 0) - U_0, \quad x = (0, \varphi) \in \partial\Omega, \tag{23}$$

$$v_i\mid_{\rho=0} = g_i - U_i, \quad x = (0, \varphi) \in \partial\Omega, \quad i = 1, 2, \cdots, \tag{24}$$

where

$$G_i = \frac{1}{i!}[\frac{\partial^i G}{\partial \varepsilon^i}]_{\varepsilon=0}; \quad G \equiv f(x, U + V, \varepsilon) - f(x, U, V, \varepsilon), \quad i = 0, 1, \cdots.$$

$$g_i = \frac{1}{i!}[\frac{\partial^i g}{\partial \varepsilon^i}]_{\varepsilon=0}, \quad i = 1, 2, \cdots,$$

and $G_i, i = 0, 1, \cdots$, are determined functions successively, which constructions are omitted.

Let the left-hand side of equations (19)-(22) be zero

$$(K_{10} + K_{20})v_i = 0, \quad i = 0, 1, \cdots. \tag{25}$$

And we have

$$(v_0)_{tt} - G_0 = 0, \tag{26}$$

$$(v_1)_{tt} - G_1 = 0, \tag{27}$$

$$(v_{i-1})_{tt} - (K_{11} + K_{21})v_{i-1} - K_{12}v_{i-3} - G_{i-1} = 0, \ i = 3, 4, \cdots. \tag{28}$$

And let

$$h_\rho = \frac{b_n}{a_{nn}},$$

That is

$$h(\rho, \varphi) = \int_0^\rho \frac{b_n(s, \varphi)}{a_{nn}(s, \varphi)} \mathrm{d}s. \tag{29}$$

Then from (25), we can obtain $v_i, i = 0, 1, \cdots$, successively, which possess boundary layer behavior

$$v_i = C_i \exp(-\delta_i \frac{\rho}{\varepsilon}), \ i = 0, 1, \cdots, \tag{30}$$

where $\delta_{i-1} \geq \delta_i > 0, \ i = 1, 2, \cdots$, are constants and C_i can be obtained from (23)–(24) and (26)–(28) successively.

Let $V_i = \psi(\rho) v_i$, where $\psi(\rho)$ is a sufficiently smooth function on $0 \leq \rho \leq \rho_0$, and satisfies $\psi(\rho) = 1$, as $0 \leq \rho \leq \frac{1}{3}\rho_0$ and $\psi(\rho) = 0$, as $\rho \geq \frac{2}{3}\rho_0$. Then we can construct the following formal asymptotic expansion of the solution u for the original problem (1)–(4):

$$u \sim \sum_{i=0}^{m} (U_i + \bar{v}_i) \varepsilon^i + O(\varepsilon^{m+1}), \ 0 < \varepsilon \ll 1. \tag{31}$$

3. The main theorem

We can prove that (31) is a uniformly valid asymptotic expansion [2].

We now get the remainder term $R(x, \varepsilon)$ the boundary value problem (1)–(4).Let

$$u(t, x, \varepsilon) = \bar{u}(t, x, \varepsilon) + R(t, x, \varepsilon), \tag{32}$$

where

$$\bar{u}(t, x, \varepsilon) \equiv \sum_{i=1}^{m} (U_i + v_i) \varepsilon^i.$$

Using the (32) We obtain

$$\begin{aligned} F(\overline{R}) =& \frac{\partial^2 R}{\partial t^2} - \varepsilon^3 L_1 R - \varepsilon L_2 R - f(t, x, \bar{u} + R, T(\bar{u} + R), \varepsilon) + f(t, x, \bar{u}, T\bar{u}, \varepsilon) \\ =& O(\varepsilon^{m+1}), x \in \Omega, \\ R =& \bar{g} = O(\varepsilon^{m+1}), x \in \partial\Omega. \\ R|_{t=0} =& \bar{h}_1 = O(\varepsilon^{m+1}), \\ \frac{\partial R}{\partial t}|_{t=0} =& \bar{h}_2 = O(\varepsilon^{m+1}). \end{aligned}$$

As to the proof of the validity of the approximation (31) it is possible to use the fixed point theorem.

The linearized differential operator L_1 reads

$$L[p] = \frac{\partial^2 p}{\partial t^2} - \varepsilon^3 L_1[p] - \varepsilon L_2[p] - f_u(t, x, \bar{u}, T\bar{u}, \varepsilon) p$$

and therefore

$$\Psi[p] \equiv F[p] - L[p] = f(t,x,\bar{u},T\bar{u},\varepsilon) - f(t,x,\bar{u}+p,T(\bar{u}+p),\varepsilon) + f_u(t,x,\bar{u}+\theta p,T(\bar{u}+p),\varepsilon)p^2, 0<\theta<1.$$

For fixed ε, the normed linear space N is chosen as

$$N = \{p | p \in C^2((0,T]\times\Omega), p|_{\partial\Omega} = \bar{g}, p|_{t=0} = \bar{h}_1, p_t|_{t=0} = \bar{h}_2\}$$

with norm

$$\|p\| = \|p\|_{L_2},$$

and the Banach space B as

$$B = \{q | q \in C((0,T]\times\Omega)\}$$

with norm

$$\|q\| = \|q\|_{L_2}.$$

From the hypotheses We may show that the condition

$$\|L^{-1}[g]\| \le l^{-1}\|g\|, \forall g \in B$$

of the fixed point theorem is fulfilled where l^{-1} is independent of ε, i.e., L^{-1} is continuous. The Lipschitz condition of the fixed point theorem becomes

$$\begin{aligned}
&\|\Psi[p_2] - \Psi[p_1]\| \\
&= \|\frac{\partial^2 f}{\partial u^2}(t,x,\bar{u}+\theta_2 p_2, T(\bar{u}+\theta_2 p_2),\varepsilon)p_2^2 - \frac{\partial^2 f}{\partial u^2}(t,x,\bar{u}+\theta_1 p_1, T(\bar{u}+\theta_1 p_1),\varepsilon)p_1^2\| \\
&= \|\frac{\partial^2 f}{\partial u^2}(t,x,\bar{u}+\theta_2 p_2, T(\bar{u}+\theta_2 p_2),\varepsilon)(p_2^2 - p_1^2) + \\
&\{\frac{\partial^2 f}{\partial u^2}(t,x,\bar{u}+\theta_2 p_2, T(\bar{u}+\theta_2 p_2),\varepsilon) - \frac{\partial^2 f}{\partial u^2}(t,x,\bar{u}+\theta_1 p_1),\varepsilon)\}p_1^2\| \\
&< C_1\|(p_1+p_2)(p_2-p_1)\| + C_1\|p_1^2(p_2-p_1)\| < Cr\|p_2-p_1\|,
\end{aligned}$$

where C_1, C_2 and C are constants indepentdent of ε and this inequality is valid for all p_1, p_2 in a ball $K_N(r)$ with $\|r\| \le 1$. Applying finally the beharior to the boundary value problem (1)-(4) we obtain the result that the remainder term uniquely exists and moreover

$$\max_{t\in(o,T],x\in\Omega} |R(t,x,\varepsilon)| = O(\varepsilon^{m+1}).$$

Thus we have the following theorem:

Theorem *Under the hypotheses $[H_1]-[H_2]$, then there exists a solution $u(t,x,\varepsilon)$ of the singulary perturbed problem (1)-(4) and holds the uniformly valid asymptotic expansion (31) for ε in $(t,x) \in [0,T_0]\times\bar{\Omega}$.*

Example Consider the following singularly perturbed problem:

$$u_{tt} - \varepsilon u_{rr} - \varepsilon u_r = -\varepsilon u^3 - 2rTu + \varepsilon[(r\sin t + \varepsilon)^3 + (2rt - \sin t)], \tag{33}$$

$$u|_{r=1} = \sin t + \varepsilon, \tag{34}$$

$$u|_{t=0} = \varepsilon, \tag{35}$$

$$u_t|_{t=0} = r + \varepsilon, \tag{36}$$

where ε is a small positive parameter and the operator T is defined by $Tu \equiv \int_0^1 u dr$. This problem satisfies hypotheses [H_1]–[H_2].

The reduced problem for the problem (33)–(36) is

$$(U_0)_{tt} = -rTU_0, \tag{37}$$

$$U_0|_{t=0} = 0, \tag{38}$$

$$(U_0)|_{t=0} = r. \tag{39}$$

It is easy to see that the solution U_0 of the problem (37)-(39) is

$$U_0 = r\sin t.$$

Let $\tau = \rho/\varepsilon$. where $\rho = 1 - r$, the boundary layer function v_0 satisfies

$$(K_{10} + K_{20})v_0 = (\partial^2/\partial\tau^2 + \partial/\partial\tau)v_0 = 0.$$

$$v_0|_{\rho=0} = -\sin t.$$

Then we obtain v_0 as

$$v_0 = -\sin t \exp\tau = -\sin t\exp(-\rho/\varepsilon) = -\sin t\exp(-(1-r)/\varepsilon).$$

We also have the solution u of the problem (33)–(36):

$$u = r\sin t + \varepsilon.$$

Let

$$u = U_0 + v_0 + R.$$

Thus from (40)-(44), we obtain

$$R = u - U_0 - v_0 = \varepsilon - \sin t\exp(-(1-r)/\varepsilon) = O(\varepsilon), \quad 0 < t < 1, \quad r \leq 1, \quad 0 < \varepsilon << 1.$$

Then we have

$$u = r\sin t - \sin t\exp(-(1-r)/\varepsilon) + O(\varepsilon), \quad 0 < t < 1, \quad r \leq 1, \quad 0 < \varepsilon << 1.$$

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非线性双曲型微分方程非局部奇摄动问题

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摘 要: 本文研究了一类具有非线性双曲型微分方程非局部奇摄动问题. 在适当的条件下，讨论了问题解的渐近性态.