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Natural Boundary Element Method for Parabolic Equations in an Unbounded Domain *

ZHU Chang-jie[1], *DU Qi-kui*[2]

(1. Huaibei Coal Industry Normal College, Anhui 235000, China;
2. School of Math. & Comp. Sci., Nanjing Normal University, Jiangsu 210097, China)

Abstract: In this paper we introduce an implementation for the efficient numerical solution of exterior initial boundary value problem for parabolic equation. The problem is reformulated as an equivalent one on a boundary Γ using natural boundary reduction. The governing equation is first discretized in time, leading to a time-stepping scheme, where an exterior elliptic problem has to be solved in each time step. By Fourier expansion, we derive a natural integral equation of the elliptic problem related to time step and Poisson integral integral formula over exterior circular domain. Finite element discretization of the natural integral equation is employed to solve this problem. The computational aspects of this method are discussed. Numerical results are presented to illustrate feasibility and efficiency of our method.

Key words: parabolic problem; natural boundary reduction; exterior problem; numerical implementation; finite element.

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1. Introduction

To solve a partial differential equation numerically in domain which extends to infinity, one commonly used method is to cut down unbounded part of spatial domain, i.e., an appropriate artificial boundary Γ_0 is introduced, and the primal problem is to restricte the computation to an appropriate large finite spatial domain $\mathcal{D}$. Then it is necessary to introduce a boundary condition on the artificial boundary which bounds $\mathcal{D}$. This naturally leads to the question " Does there exist an artificial boundary condition such that the numerical solution of the primal problem in $\mathcal{D}$, with this boundary condition, coincides exactly with the restriction to $\mathcal{D}$ of the solution in the unbounded domain?" The question has been answered affirmatively, such as the coupled method based on boundary element method(see, e.g.[7], [8]). We here shall concentrate on considering the problem in original

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Biography: ZHU Chang-jie (1964-), male, M.Sc., Associate Professor.

domain. Since boundary element methods can convert the problem in domain to integral equation on the boundary of the domain, such as classical boundary element methods. The natural boundary element methods initiated and developed by Feng Kang and Yu Dehao in the late seventies(see, e.g. [1 − 9]) have been shown to possess good computational properties and to be very effective in practice. They have some distinctive advantages comparing with the classical boundary element methods. For example, it is easy to be implemented on the computing, it has good stability of the numerical results, it is fully compatible with finite element method, and it can be coupled with finite element method naturally and directly. For general linear elliptic problems, the theory of natural boundary element method is being perfected(see, e.g. [9]). But up to now, natural boundary element method has not been applied to solve evolution equations yet, such as heat conduction problems, wave equations, etc. Mathematics techniques used for elliptic problems can not be applied to the time-dependent problem directly. Therefore, we must explore some new approaches. Motivated by the above reasons, we make an attempt in this field. In this paper a new and effectual method is given and natural boundary reduction for the parabolic equations is realized.

Let Ω be a simple connected domain in R^2 with smooth boundary $\Gamma := \partial\Omega$, $\Omega^c := R^2\backslash\overline{\Omega}$. For any fixed positive real number T, we write $J := (0, T]$. Now considering the following initial boundary value problem:

$$\begin{cases} u_t - \Delta u = f(x,t), & (x,t) \in \Omega^c \times J, \\ \frac{\partial u}{\partial n} = g(x,t), & (x,t) \in \Gamma \times J, \\ u(x,0) = u_0(x), & x \in \Omega^c. \end{cases} \tag{1.1}$$

Here $u(x,t)$ is the unknown function, u_t denotes derivative with respect to time t, and $f(x,t)$, $g(x,t)$ and $u_0(x)$ are given functions, which satisfy appropriate conditions. $\frac{\partial}{\partial n}$ is the normal derivative operator on Γ (n is the unit normal vector on boundary Γ of domain Ω^c toward the interior of domain Ω). Futhermore, we assume the function $u(x,t)$ is bounded at infinity. However, there is no need in a "radiation condition" at infinity [10, 11].

2. Discrete problem in time.

According to the theory of natural boundary element methods[9], it is major work for us to derive the expression of natural integral operator $\mathcal{K}_\lambda$, and to realize its numerical solution. Since the expressions of natural integral operator $\mathcal{K}_\lambda$ and Poisson integral operator $\mathcal{P}$ depend on geometric figure of domain, we assume that domain Ω is a circular domain of radius R, centered at the origin. For the sake of convenience, we also assume that the solution considered the problem has appropriate smooth.

Let τ be the time-step interval, and write $t_k = k \cdot \tau$, $u^k(x) = u(x, t_k)$, $z^k(x) = u_t(x, t_k)$. Taylor expansion applied to equation (1.1) consists of the following two equations

$$z^k - \Delta u^k = f^k, \tag{2.1}$$

$$u^k = u^{k-1} + \tau\{(1-\gamma)z^{k-1} + \gamma z^k\}. \tag{2.2}$$

Here $\gamma \in (0,1]$ and $k = 1, 2, \cdots, [T/\tau]$. We define

$$\lambda := (\sqrt{\tau\gamma})^{-1}, \quad \tilde{u}^k := u^{k-1} + \tau(1-\gamma)z^{k-1}, \quad \tilde{f}^k := -\tilde{u}^k - \tau\gamma f^k.$$

After some algebra, equations (2.1) – (2.2) yield the following formulae

$$\Delta u^k - \lambda^2 u^k = \lambda^2 \tilde{f}^k, \quad x \in \Omega^c, \tag{2.3}$$

$$z^k = \lambda^2(u^k - \tilde{u}^k). \tag{2.4}$$

We can summarize the solution procedure of the above discrete problem as follows:

I) Prodiction

$$\tilde{u}^k = u^{k-1} + \tau(1-\gamma)z^{k-1}. \tag{2.5}$$

II) Solve the problem

$$\Delta u^k - \lambda^2 u^k = \lambda^2 \tilde{f}^k, \qquad x \in \Omega^c, \tag{2.6}$$

$$\frac{\partial u^k}{\partial n} = g^k, \quad x \in \Gamma, \tag{2.7}$$

$$|u^k| < +\infty, \qquad |x| \longrightarrow +\infty. \tag{2.8}$$

III) Update value

$$z^k = \lambda^2(u^k - \tilde{u}^k). \tag{2.9}$$

From the theory of natural boundary reduction, the Dirichlet boundary value u_0^k and Neumann boundary value $\frac{\partial u^k}{\partial n}$ satisfy the following relation

$$\frac{\partial u^k}{\partial n} + N(\lambda, R; \tilde{f}^k) = \mathcal{K}_\lambda \cdot u_0^k, \tag{2.10}$$

the solution u^k and its Dirichlet boundary value u_0^k is given as follows:

$$u^k = \mathcal{P}_\lambda \cdot u_0^k + F(\lambda, R; \tilde{f}^k, r, \theta), \tag{2.11}$$

where $\mathcal{K}_\lambda$ and $\mathcal{P}_\lambda$ are known as natural integral operator and Poisson integral operator, respectively. Equations (2.10) and (2.11) are usually known as natural integral equation and Poisson integral formula, respectively. Equations (2.10) and (2.11) are equivalent to the problems (2.6)-(2.8). In addition, equation (2.10) is equivalent to the following variational problem:

$$\begin{cases} \text{Find } \ u^k \in H^{1/2}(\Gamma) \text{ such that} \\ b(u^k, v^k) = \langle \frac{\partial u^k}{\partial n} + N(\lambda, R; \tilde{f}^k), v^k \rangle, \quad \forall v^k \in H^{1/2}(\Gamma), \end{cases} \tag{2.12}$$

where

$$b(u^k, v^k) := \langle \mathcal{K}_\lambda u_0^k, v^k \rangle \equiv \int_\Gamma (\mathcal{K}_\lambda u_0^k) \cdot v^k \mathrm{d}S, \quad \langle w, \mu \rangle := \int_\Gamma w\mu \mathrm{d}S. \tag{2.13}$$

The meanings of some notations can be seen in the next text. The expressions of natural integral equation and Poisson integral formula will be given in Section 3.

3. Natural boundary reduction

Boundary Γ, which is a circle of radius R, centered at the origin, can be expressed as follows:$\Gamma = \{(r,\theta) \mid r = R,\ \theta \in [0, 2\pi]\}$. While the direction of the unit outward normal to Γ is opposite to the direction of r, i.e. $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$. By Fourier expansion, the solution of equations (2.6)-(2.8) in the polar coordinates can be expressed with the following form:

$$u^k(r,\theta) = \frac{1}{2}a_0(r) + \sum_{n=1}^{+\infty}[a_n(r)\cos n\theta + b_n(r)\sin n\theta]. \tag{3.1}$$

It is not difficult to get the solution $u^k(r,\theta)$ of the equations (2.6)–(2.8) as follows:

$$u^k(r,\theta) = \frac{1}{2\pi}\sum_{n=0}^{+\infty}\varepsilon_n\frac{K_n(\lambda r)}{K_n(\lambda R)}\int_0^{2\pi}\cos n(\theta-\theta')\cdot u^k(R,\theta')\mathrm{d}\theta' + F(\lambda,R;\tilde{f}^k,r,\theta), \quad r > R \tag{3.2}$$

where

$$F(\lambda,R;\tilde{f}^k,r,\theta) = \frac{\lambda^2}{2}\sum_{n=0}^{+\infty}\varepsilon_n\int_R^{+\infty}\sigma^2 G_n(r,\sigma)[\tilde{f}_n^{k,c}(\sigma)\cos n\theta + \tilde{f}_n^{k,s}(\sigma)\sin n\theta]\mathrm{d}\sigma, \tag{3.3}$$

$$\sigma^2 G_n(r,\sigma) = \begin{cases} \phi_n(\sigma)\cdot\psi_n(r)/E_n(\sigma), & r \le \sigma, \\ \psi_n(\sigma)\cdot\phi_n(r)/E_n(\sigma), & r \ge \sigma, \end{cases} \tag{3.4}$$

$$\phi_n(\sigma) = K_n(\lambda\sigma), \tag{3.5}$$

$$\psi_n(\sigma) = I_n(\lambda\sigma)\phi_n(R) - \phi_n(\sigma)I_n(\lambda R), \tag{3.6}$$

$$E_n(\sigma) = \psi_n(\sigma)\phi_n'(\sigma) - \phi_n(\sigma)\psi_n'(\sigma), \tag{3.7}$$

$$\tilde{f}_n^{k,c}(\sigma) = \frac{1}{\pi}\int_0^{2\pi}\cos n\theta\cdot\tilde{f}^k(\sigma,\theta)\mathrm{d}\theta,$$

$$\tilde{f}_n^{k,s}(\sigma) = \frac{1}{\pi}\int_0^{2\pi}\sin n\theta\cdot\tilde{f}^k(\sigma,\theta)\mathrm{d}\theta,$$

where $I_n(x)$ and $K_n(x)$ are respectively modified Bessel functions of the first and second kind, $n = 0, 1, 2, \cdots$. And $\varepsilon_n = 1, n = 0$; $\varepsilon_n = 2, n > 0$. We now differentiate both sides of (3.2) with respect to r, and take limit as r approaches $R+0$, we get the boundary integral equation of the Neumann problem in Ω^c

$$\frac{\partial u^k(R,\theta)}{\partial n} = \frac{\lambda}{2\pi}\sum_{n=0}^{+\infty}\varepsilon_n\frac{K_n'(\lambda r)}{K_n(\lambda R)}\int_0^{2\pi}\cos(\theta-\theta')\cdot u^k(R,\theta')\mathrm{d}\theta - N(\lambda,R;\tilde{f}^k,\theta), \tag{3.8}$$

where

$$N(\lambda,R;\tilde{f}^k,\theta) = \frac{\lambda^2}{2}\sum_{n=0}^{+\infty}\varepsilon_n\int_R^{+\infty}\overline{G}_n(\lambda,R;\sigma)\cdot[\tilde{f}_n^{k,c}(\sigma)\cos n\theta + \tilde{f}_n^{k,s}(\sigma)\sin n\theta]\mathrm{d}\sigma, \tag{3.9}$$

$$\overline{G}_n(\lambda,R;\sigma) = -\frac{K_n(\lambda\sigma)}{K_n(\lambda R)}\cdot\frac{\sigma}{R}, \quad n=0,1,2,\cdots. \tag{3.10}$$

Equations (3.2) and (3.8) are respectively Poisson integral formula and natural integral equation. If we obtain function $u^k(R,\theta)$ by narural integral equation (3.8), we can get the solution of the original initial boundary value problem (1.1) by Poisson integral formula (3.2). But the solution of the problem can be obtained directly by Poisson integral formula (3.2) for Cauchy-Dirichlet initial boundary value problem, because the function $u^k(R,\theta)$ is given.

4. Direct investigation of natural integral operator

We now investigate some properties of natural integral operator. To this end, we will need some preliminaries. For all $f \in H^p(\Gamma)$, $f(R,\theta)$ can be expressed as follows(in the sense of mean convergence).

$$f(R,\theta) = \sum_{-\infty}^{+\infty} f_n \cdot e^{in\theta},$$

where $f_n (n \in Z)$ is the coefficients of Fourier series. We introduce the $H^p(\Gamma)-$norm of function $f(R,\theta)$ as follows

$$\|f\|_{p,\Gamma} := [\sum_{-\infty}^{+\infty}(1+n^2)^p \cdot |f_n|^2]^{1/2}. \tag{4.1}$$

Lemma 4.1[9] *Let* $u = \sum_{-\infty}^{+\infty} u_n \cdot e^{in\theta}$, $v = \sum_{-\infty}^{+\infty} v_n \cdot e^{in\theta}$, *then* $u * v = \sum_{-\infty}^{+\infty}(2\pi u_n v_n)\cdot e^{in\theta}$. *Where* $*$ *denotes the convolution with respect to variable* θ. □

If we define $h_\ell(x) := \frac{K_\ell'(x)}{K_\ell(\lambda R)}$, then natural integral operator $\mathcal{K}_\lambda$ can be expressed by Lemma 4.1 as follows

$$\mathcal{K}_\lambda u = -\lambda \sum_{-\infty}^{+\infty} u_n h_n(\lambda R)\cdot e^{in\theta}. \tag{4.2}$$

Lemma 4.2 *For the modified Bessel function* $K_n(x)$, $(x > 0)$, *the following assertion holds*

$$|\frac{K_n'(x)}{K_n(x)}|^2 \cdot \frac{1}{1+n^2} = O(1), \quad n \longrightarrow +\infty. \tag{4.3}$$

Proof From [12], the Hankel function $H_n^{(1)}(x)$ of the first kind has the following asymptotic formula for large index

$$H_n^{(1)}(x) = -i\sqrt{\frac{2}{n\pi}}(\frac{2n}{ex})^n\{1+O(\frac{1}{n})\}, \quad n \longrightarrow +\infty,$$

and $K_n(x)$ satisfies

$$K_n(x) = \frac{\pi i}{2} e^{i\frac{n\pi}{2}} H_n^{(1)}(ix).$$

Thus, we can derive immediately

$$K_n(x) = \sqrt{\frac{\pi}{2n}}(\frac{2n}{ex})^n\{1+O(\frac{1}{n})\}, \quad n \longrightarrow +\infty \tag{4.4}$$

as well as the recursion formula(see [12])

$$K_n'(x) = -\frac{1}{2}[K_{n+1}(x) + K_{n-1}(x)], \quad (n \geq 1); \quad K_0'(x) = -K_1(x). \tag{4.5}$$

By (4.5), the expression we wish to bound becomes

$$\frac{K_n'(x)}{K_n(x)} = -\frac{1}{2}\{\frac{K_{n+1}(x)}{K_n(x)} + \frac{K_{n-1}(x)}{K_n(x)}\}, \quad (n \geq 1). \tag{4.6}$$

Using (4.4) in (4.6), we see the first term is $O(n)$, and the second term tends zero as n approaches $+\infty$, yielding

$$|\frac{K_n'(x)}{K_n(x)}|^2 \cdot \frac{1}{1+n^2} = O(n^2) \cdot \frac{1}{1+n^2} = O(1), \quad n \longrightarrow +\infty.$$

Theorem 4.3 *For $p \geq 0$, natural integral operator $\mathcal{K}_\lambda$ is a bounded linear operator from $H^{p+1/2}(\Gamma)$ to $H^{p-1/2}(\Gamma)$, i.e. there exists a positive constant C such that*

$$||\mathcal{K}_\lambda f||_{p-1/2,\Gamma} \leq C \cdot ||f||_{p+1/2,\Gamma}, \quad \forall f \in H^{p+1/2}(\Gamma). \tag{4.7}$$

Proof Obviously, natural integral operator $\mathcal{K}_\lambda$ is a linear operator on $H^{p+1/2}(\Gamma)$. For any $f \in H^{p+1/2}(\Gamma), (p \geq 0)$, setting $F := \mathcal{K}_\lambda f$. From (4.2) we have $F = \sum_{-\infty}^{+\infty} F_n \cdot e^{in\theta}$. where $F_n = -\lambda f_n h_n(\lambda R)$. From Lemma 4.2, for any $n \in Z$, the following inequality holds

$$|h_n(\lambda R)|^2 \cdot \frac{1}{1+n^2} \leq C.$$

Thus

$$\begin{aligned} ||\mathcal{K}_\lambda f||^2_{p-1/2,\Gamma} \equiv ||F||^2_{p-1/2,\Gamma} &= \sum_{-\infty}^{+\infty}(1+n^2)^{p-1/2} \cdot |F_n|^2 \\ &= \lambda^2 \sum_{-\infty}^{+\infty}(1+n^2)^{p+1/2} \cdot |f_n|^2 \cdot (|h_n(\lambda R)|^2 \cdot \frac{1}{1+n^2}) \\ &\leq C\lambda^2 \sum_{-\infty}^{+\infty}(1+n^2)^{p+1/2} \cdot |f_n|^2 \\ &= C_\lambda \cdot ||f||^2_{p+1/2,\Gamma}. \end{aligned}$$

Theorem 4.4 *The bilinear form $b(\cdot,\cdot)$ induced by natural integral operator $\mathcal{K}_\lambda$ is a symmetric and continuous on $H^{1/2}(\Gamma) \times H^{1/2}(\Gamma)$ and non-negative definite on $H^{1/2}(\Gamma)$, i.e.,*

(i) $b(u,v) = b(v,u), \forall u, v \in H^{1/2}(\Gamma)$;

(ii) There exists a positive constant C such that

$$b(u,v) \leq C \cdot ||u||_{1/2,\Gamma} \cdot ||v||_{1/2,\Gamma}, \quad \forall u, v \in H^{1/2}(\Gamma);$$

(iii) $b(u,u) \geq 0, \forall u \in H^{1/2}(\Gamma)$.

Proof (i) From Lemma 4.1, for any $u \in H^{1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$, since

$$\begin{aligned} b(u,v) = \langle \mathcal{K}_\lambda u, v\rangle &= \int_\Gamma (\sum_{-\infty}^{+\infty} v_n \cdot e^{in\theta}) \cdot (\mathcal{K}_\lambda u)\mathrm{d}S \\ &= -\lambda R \int_0^{2\pi} (\sum_{-\infty}^{+\infty} v_n \cdot e^{in\theta}) \cdot (\sum_{-\infty}^{+\infty} u_m \cdot h_m(\lambda R) \cdot e^{im\theta})\mathrm{d}\theta \\ &= -\lambda R \cdot 2\pi \sum_{-\infty}^{+\infty} h_n(\lambda R) \cdot u_n v_n, \end{aligned}$$

so

$$b(v,u) = -\lambda R \cdot 2\pi \sum_{-\infty}^{+\infty} h_n(\lambda R) \cdot v_n u_n = b(u,v);$$

(ii) By Theorem 4.3., for any $u \in H^{1/2}(\Gamma)$ and $v \in H^{1/2}(\Gamma)$,

$$b(u,v) = \langle \mathcal{K}_\lambda u, v\rangle \le \|\mathcal{K}_\lambda u\|_{-1/2,\Gamma} \cdot \|v\|_{1/2,\Gamma} \le C_\lambda \|u\|_{1/2,\Gamma} \cdot \|v\|_{1/2,\Gamma};$$

(iii) For any $u \in H^{1/2}(\Gamma)$, by (i) we have

$$b(u,u) = -\lambda R \cdot 2\pi \sum_{-\infty}^{+\infty} h_n(\lambda R) \cdot |u_n|^2.$$

Since $K_n(x) > 0$ for $x > 0$, so we get $K_n'(x) < 0$ by (4.5). We get $h_n(x) < 0$ for $x > 0$. Which proves last assertion. □

5. Numerical implementation of natural integral equation.

Now we partition the circumference Γ into some finite elements, which satisfies usual regular conditions. For simplicity, we take uniform subdivision. Now let $S_h(\Gamma)$ be the finite element subspace of space $H^{1/2}(\Gamma)$. So we can obtain the approximate variational problem of the problem (2.12) as follows:

$$\begin{cases} \text{Find } u_h^k \in S_h(\Gamma) \text{ such that} \\ b(u_h^k, v^k) = \langle g^k + N(\lambda, R; \tilde{f}^k, \theta), v^k\rangle, \quad \forall v^k \in S_h(\Gamma) \end{cases} \tag{5.1}$$

If we take $S_h(\Gamma) = \mathrm{span}\{\psi_1(\theta), \psi_2(\theta), \cdots, \psi_M(\theta)\}$, then we get the following system of algebaric equations of problem (5.1)

$$Q \cdot U^k = b^k \tag{5.2}$$

where

$$Q := (q_{ij})_{M\times M}; U^k := (u_1^k, u_2^k, \cdots, u_M^k)^T, b^k := (b_1^k, b_2^k, \cdots, b_M^k)^T,$$

$$b_j^k = \int_0^{2\pi} (g^k(R,\theta) + N(\lambda, R; \tilde{f}^k, \theta)) \cdot \psi_j(\theta)\mathrm{d}\theta,$$

$$q_{ij} = -\frac{\lambda}{2\pi}\sum_{n=0}^{+\infty}\varepsilon_n\frac{K_n'(\lambda R)}{K_n(\lambda R)}\int_0^{2\pi}\int_0^{2\pi}\cos n(\theta-\theta')\psi_i(\theta)\psi_j(\theta')\mathrm{d}\theta\mathrm{d}\theta'.$$

5.1. Calculation of the elements of the stiffness matrix.

5.1.1. Piecewise linear basis functions

Taking

$$L_i(\theta) = \begin{cases} \frac{N}{2\pi}(\theta-\theta_{i-1}), & \theta\in[\theta_{i-1},\theta_i] \\ \frac{N}{2\pi}(\theta_{i+1}-\theta), & \theta\in[\theta_i,\theta_{i+1}] \\ 0, & \text{otherwise,} \end{cases} \tag{5.3}$$

where $i = 1,2,\cdots,N$; $\theta_i = \frac{i}{N}2\pi$. $L_i(\theta_j) = \delta_i^j = \begin{cases} 1, & i=j, \\ 0, & i\neq j, \end{cases}$, $i,j = 1,2,\cdots,N$; $\sum_{i=1}^N L_i(\theta) = 1$. and $\mathrm{span}\{L_i(\theta)\}_{i=1}^N \subset H^1(\Gamma)\subset H^{1/2}(\Gamma)$. It is not difficult to obtain the following results

$$q_{ij} = -\frac{2\pi\lambda}{N^2}\{\frac{K_0'(\lambda R)}{K_0(\lambda R)} + \frac{2N^4}{\pi^4}\sum_{n=1}^{+\infty}\frac{1}{n^4}\frac{K_n'(\lambda R)}{K_n(\lambda R)}\sin^4(\frac{n\pi}{N})\cdot\cos[\frac{(i-j)}{N}\cdot 2n\pi]\}. \tag{5.4}$$

Obviously, $q_{ij} = q_{ji}$, $i,j = 1,2,\cdots,N$. Setting

$$a_m = -\frac{2\pi\lambda}{N^2}\{\frac{K_0'(\lambda R)}{K_0(\lambda R)} + \frac{2N^4}{\pi^4}\sum_{n=1}^{+\infty}\frac{1}{n^4}\frac{K_n'(\lambda R)}{K_n(\lambda R)}\sin^4(\frac{n\pi}{N})\cdot\cos(\frac{nm}{N}\cdot 2\pi)\},$$

$$m = 0,1,\cdots,N-1. \tag{5.5}$$

The series a_m is convergent(See Theorem 5.1). Thus

$$q_{ij} = a_{|i-j|} = q_{ji}, \quad i,j = 1,2,\cdots,N, \tag{5.6}$$

$$Q = (a_{|i-j|})_{N\times N} = ((a_0,a_1,\cdots,a_{N-1})). \tag{5.7}$$

The stiffness matrix Q is a cyclical matrix producted by $a_0,a_1,\cdots,a_{N-1}$.

$$Q = \begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & a_1 & \cdots & a_{N-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{bmatrix}_{N\times N}.$$

Note that $a_i = a_{N-i}(i = 0,1,2,\cdots,N-1)$, so we only calculate $[\frac{N}{2}]+1$ elements, and can get the matrix Q. It is easy to be implemented on calculation and storage.

The approximate expression of Poisson integral formula (3.2) as follows:

$$u_h^k(r,\theta) = \frac{1}{N}\sum_{j=1}^N u_j^k\cdot\{\frac{K_0(\lambda r)}{K_0(\lambda R)} + \frac{2N^2}{\pi^2}\sum_{n=1}^{+\infty}[\frac{1}{n^2}\frac{K_n(\lambda r)}{K_n(\lambda R)}(\sin\frac{n\pi}{N})^2\cdot\cos n(\theta-j\frac{2\pi}{N})]\}+$$

$$F(\lambda,R;\tilde{f}^k,r,\theta), \quad r>R. \tag{5.8}$$

5.1.2. Piecewise quadratic basis functions
We take

$$\varphi_{2i-1}(\theta)=\begin{cases}(\frac{N}{\pi})^2(\theta-\theta_{2i-2})\cdot(\theta_{2i}-\theta), & \theta\in[\theta_{2i-2},\theta_{2i}],\\ 0, & \text{otherwise.}\end{cases} \tag{5.9}$$

$$\varphi_{2i}(\theta)=\begin{cases}\frac{1}{2}(\frac{N}{\pi})^2(\theta-\theta_{2i-1})(\theta-\theta_{2i-2}), & \theta\in[\theta_{2i-2},\theta_{2i}],\\ \frac{1}{2}(\frac{N}{\pi})^2(\theta-\theta_{2i+1})(\theta-\theta_{2i+2}), & \theta\in[\theta_{2i},\theta_{2i+2}],\\ 0, & \text{otherwise,}\end{cases} \tag{5.10}$$

$$i=1,2,\cdots,N,$$

satisfying $\varphi_k(\theta_j)=\delta_k^j, k,j=1,2,\cdots,2N$. where $\theta_i=\frac{i}{N}\cdot\pi$, $i=1,2,\cdots,2N$. We know easy span$\{\varphi_1(\theta),\varphi_2(\theta),\cdots,\varphi_{2N}(\theta)\}\subset H^1(\Gamma)\subset H^{1/2}(\Gamma)$, and $\sum_{j=1}^{2N}\varphi_j(\theta)=1$. It is not difficult to obtain the following relations

$$q_{2i,2j}=b_{|i-j|};\quad q_{2i-1,2j-1}=c_{|i-j|};\quad q_{2j,2i-1}=d_{i-j-1}\ (i\geq j,\ d_{-1}=d_0).$$

where

$$\begin{cases}b_m=-\frac{\lambda h^2}{9\pi}\{2\frac{K_0'(\lambda R)}{K_0(\lambda R)}+\frac{9}{h^4}\sum_{n=1}^{+\infty}[\frac{1}{n^4}\frac{K_n'(\lambda R)}{K_n(\lambda R)}(\frac{2}{nh}\sin 2nh-\cos 2nh\\ \qquad -3)^2\cdot\cos 2mnh]\},\\ c_m=-\frac{8\lambda h^2}{9\pi}\{\frac{K_0'(\lambda R)}{K_0(\lambda R)}+\frac{18}{h^4}\sum_{n=1}^{+\infty}[\frac{1}{n^4}\frac{K_n'(\lambda R)}{K_n(\lambda R)}(\frac{1}{nh}\sin nh-\\ \qquad \cos nh)^2\cdot\cos 2mnh]\},\\ d_m=-\frac{4\lambda h^2}{9\pi}\{\frac{K_0'(\lambda R)}{K_0(\lambda R)}-\frac{9}{h^4}\sum_{n=1}^{+\infty}[\frac{1}{n^4}\frac{K_n'(\lambda R)}{K_n(\lambda R)}(\frac{1}{nh}\sin nh-\cos nh)\\ \qquad \cdot(\frac{2}{nh}\sin 2nh-\cos 2nh-3)\cdot\cos 2(m+\frac{1}{2})nh]\},\\ \qquad m=0,1,2,\cdots,N-1.\end{cases} \tag{5.11}$$

Note that $b_{N-j}=b_j$, $c_{N-j}=c_j$, $d_{N-j}=d_{j-1}$ $(j=0,1,2,\cdots,N-1)$. We only calculate

$$b_0,\ b_1,\cdots,\ b_{[\frac{N}{2}]};\quad c_0,\ c_1,\cdots,\ c_{[\frac{N}{2}]};\quad d_0,\ d_1,\cdots,\ d_{[\frac{N-1}{2}]}.$$

and can obtain matrix Q. It is easy to be implemented on calculation and storage.

The approximate expression of Poisson integral formula (3.2) as follows:

$$\begin{aligned}u_h^k(r,\theta)=&\frac{h}{\pi}\sum_{j=1}^{N}\{[\frac{2}{3}\frac{K_0(\lambda r)}{K_0(\lambda R)}+\frac{4}{h^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}\frac{K_n(\lambda r)}{K_n(\lambda R)}(\frac{1}{nh}\sin nh-\\ &\cos nh)\cdot\cos n(\theta-(2j-1)h)]\cdot u_{2j-1}^k+[\frac{1}{3}\frac{K_0(\lambda r)}{K_0(\lambda R)}-\\ &\frac{1}{h^2}\sum_{n=1}^{+\infty}\frac{1}{n^2}\frac{K_n(\lambda r)}{K_n(\lambda R)}\cdot(\frac{2}{nh}\sin 2nh-\cos 2nh-3)\cdot\cos n(\theta-\\ &2jh)]\cdot u_{2j}^k\}+F(\lambda,R;\tilde{f}^k,r,\theta),\quad r>R.\end{aligned} \tag{5.12}$$

Remark 5.1 The stiffness matrix of the system of linear algebraic equations obtained by the above quadratic basis functions does not use the results obtained by linear basis functions. Of course, we can construct quadratic basis functions based on linear basis functions, so that we can make best use of the results obtained by linear basis functions. We can refer to [9].

Theorem 5.1 *The series a_m, b_m, c_m and d_m are all absolute convergent.*

Proof From Lemma 4.1., the following inequality holds

$$|\frac{K_n'(\lambda R)}{K_n(\lambda R)}|^2 \cdot \frac{1}{(1+n^2)} \leq C.$$

So, we see that the series a_m, b_m, c_m and d_m are convergent like $\sum n^{-3}$, which proves our assertion. □

5.2. Numerical solution procedure. From the above statement, the numerical solution procedure is summarized as follows:

- For time-step τ, loop u^k : $k = 1, 2, \cdots\cdots$

  Step 1. Compute the predicted value $\tilde{u}^k$ by using (2.5).

  Step 2. $u^{k,(0)} \Longleftarrow \tilde{u}^k$.

  - Loop $u^{k,(i)}$: $i = 1, 2, \cdots\cdots$

    Step 3. Find $N(\lambda, R; \tilde{f}^k, \theta)$ by using (3.9).

    * Iteration: Solve the system of algebraic equations to obtain U^k :

      Step 4. Solve the system of algebraic equations by using (5.2) to obtain solutions $U^{k,(j)}$: $j = 1, 2, \cdots\cdots$.

      Step 5. Check the convergence of $U^{k,(j)}$. If converged, $U^k \Longleftarrow U^{k,(j)}$, go to Step 6; else continue.

    * Next j.

    Step 6. Find $F(\lambda, R; \tilde{f}^k, r, \theta)$ by using (3.3).

    Step 7. Find $u^{k,(i)}$ by using (5.8) or (5.12).

    Step 8. Check $u^{k,(i)}$: If it satisfies conditions given, $u^k \Longleftarrow u^{k,(i)}$, go to Step 9; else continue.

  - Next i

  Step 9. Update z^k by using (2.9).

- Next time level loop.

Since the problem considered is time-dependent, we take the time-step loop iteration to solve u^k ($k = 1, 2, \cdots\cdots$) for each time t_k. The outermost loop in the solution procedure just complete the procedure. Of course, if the problem is elliptic, there is no need the loop. We now make some comments on the computational procedure.

Remark 5.2 The algebraic system (5.2) is solved by using many methods, but it had better use iteration methods so that we make best use of the property of the cyclical matrix, and decrease storage.

Remark 5.3 The computational procedure involves the computation of the integrals $N(\lambda, R; \tilde{f}^k, \theta)$ and $F(\lambda, R; \tilde{f}^k, r, \theta)$. We may restrict $R \le r < +\infty$ to $R \le r \le r_{max}$. They are calculated numerically by using a simple trapezoidal rule per integration cell in both the r−and−θ directions. At the same time, the expressions for $N(\lambda, R; \tilde{f}^k, \theta)$ and $F(\lambda, R; \tilde{f}^k, r, \theta)$ involve infinite series. In practice all the infinite sums are truncated after a finite number of terms, M_1. So are the expressions (5.5), (5.8), (5.11) and (5.12).

6. Numerical examples.

Now we present a few numerical results for the problem considered. Considering the problem with exterior unit circle domain as follows. Taking $f(x,t) = \{-\frac{1}{r^3}\sin(\frac{3}{2}\pi r) + \frac{3\pi}{2}\frac{1}{r^2}\cos(\frac{3}{2}\pi r)\} \cdot e^{-(\frac{3\pi}{2})^2 t}$. Where $r = \sqrt{x_1^2 + x_2^2}$. Functions $g(x,t) = -e^{-(\frac{3\pi}{2})^2 t}$, $u_0(x) = -(\frac{3\pi}{2})^2 \cdot \frac{1}{r}\sin(\frac{3}{2}\pi r)$. We calculate the approximate solutions $u^k(1,\theta)$ of natural integral equation by using the above methods. We substitute $\sum_{n=1}^{M_1}$ for $\sum_{n=1}^{+\infty}$ in the expressions of the elements of the stiffness matrix and integrals $N(\lambda, 1; \tilde{f}^k, \theta)$ and $F(\lambda, 1; \tilde{f}^k, r, \theta)$. *Error* denotes maximum of the relative error. The computational results are as follows.

TABLE 6.1. Linear element, $\gamma = 1$, $t = 0.2$

M	M_1	error		
		$\tau = 0.05$	$\tau = 0.025$	$\tau = 0.0125$
8	20	13.876254	12.102372	10.104215
16	40	4.154872	3.522475	3.112512
32	80	1.627335	1.102134	1.001237
64	120	0.532244	0.322768	0.299825

TABLE 6.1. Quadratic element, $\gamma = 1$, $t = 0.2$

M	M_1	*Error*		
		$\tau = 0.05$	$\tau = 0.025$	$\tau = 0.0125$
8	20	12.345236	10.110225	7.958762
16	40	3.221164	3.000563	2.243254
32	80	0.895462	0.815463	0.680512
64	120	0.293426	0.224532	0.202754

The numerical results above show that natural boundary element method is very effectual. But we see that it is no quite evident that the numerical results by using quadratic element is better than the those by using linear element. This is different from the elliptic problems. The main reason may be the numerical integrations and the choice of time-step, because the error must be composed of time-step and interpolation.

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References:

[1] FANG Kang. *Differential versus integral equations and finite versus infinite elements* [J]. Mathematicae Numerica Sinica(in Chinese), 1980, **2**(1): 100–105.

[2] FANG Kang. *Canonical boundary reduction and finite element method* [C]. Proceedings of International Invitational Symposium on the Finite Element Method (1981,Heifei), Science Press, Beijing, 1982, 330–352.

[3] FANG Kang, YU De-hao. *Canonical integral equations of elliptic boundary value problems and their numerical solutions* [C]. Proceedings of China-France Symposium on the Finite Element Method (1982, Beijing), Science Press, Beijing, 1983, 211-252.

[4] FANG Kang. *Finite element method and natural boundary reduction* [C]. Proceedings of the International Congress of Mathematicians, Warszawa, 1983, 1439-1453.

[5] YU De-hao. *Canonical integral equations of biharmonic elliptic boundary value problems* [J]. Mathematicae Numerica Sinica(in Chinese), 1982, 4(3): 330-336.

[6] YU De-hao. *Numerical solutions of harmonic and biharmonic canonical integral equations in interior or exterior circlular* [J]. J. Comp. Math., 1983, 1(1): 52-62.

[7] YU De-hao. *Coupling canonical boundary element method with FEM to solve harmonic problem over cracked domain* [J]. J. Comp. Math., 1983, 1(3):195-202.

[8] YU De-hao. *A direct natural coupling of BEM and FEM* [C]. Boundary Elements XIII, C.A.Brebbia, G.S.Gipson eds., Computational Mechanics Publications, Southampton, 1991, 995-1004.

[9] YU De-hao. *Mathematical Theory of Natural Boundary Element Method* [M], Pure Mathematics and Applied Mathematics, No.26, Science Press(in Chinese), Beijing, 1993.

[10] COURANT R, HILBERT D. *Methods of Mathematics Physics* [M] (translated by Qian Min and Guo Dunren), Science Press(in Chinese), Beijing, 1981.

[11] COURANT R, HILBERT D. *Methods of Mathematics Physics* [M]. Vol.II, Wiley, NewYork, 1962.

[12] LIU Shi-kuo, LIU Shi-da. *Special Functions* [M]. Atmospheric Phenomena Press(in Chinese), Beijing, 1988.5.

无界区域抛物方程自然边界元方法

朱昌杰[1], 杜其奎[2]

(1. 淮北煤炭师范学院，安徽 淮北 235000;

2. 南京师范大学数学与计算机科学学院，江苏 南京 210097)

摘 要: 本文应用自然边界元方法求解无界区域抛物型初边值问题。首先将控制方程对时间进行离散化，得到关于时间步长离散化的椭圆型问题。通过 Fourier 展开，导出相应问题的自然积分方程和 Poisson 积分公式。研究了自然积分算子的性质，并讨论了自然积分方程的数值解法，最后给出数值例子。从而解决了抛物型问题的自然边界归化和自然边界元方法。