Journal of Mathematical Research & Exposition
Vol.22, No.2, 197-204, May, 2002

The Continuous Selections for the Metric Projection *

SONG Wen-hua
(Inst. of Appl. Math., Hebei University of Economics and Trade, Shijiazhuang 050091, China)

Abstract: Some results concerning weakly continuous selection for set-valued mapping are given and, applied to metric projection. Let Y be a subspace of a Banach space X. If Y is a separable reflexive Banach space, removed a first category subset, the metric projection is weakly lower semicontinuous and admits a weakly continuous selection.

Key words: continuous selection; metric projection; weakly continuous

Classification: AMS(2000) 41A65/CLC O174.41

Document code: A **Article ID:** 1000-341X(2002)02-0197-08

1. Introduction

Let Y be a subset of a normed linear space X. The metric projection onto Y is the set-valued mapping

$$P_Y(x) = \{y \in Y; \|x-y\| = d(x,Y)\},$$

where $d(x,Y) = \inf\{\|x-y\|; y \in Y\}$. If, for each $x \in X$, $P_Y(x) \neq \emptyset$, then Y is called proximinal. One of the most interesting and important problem in metric projection is the existence of continuous selection. A continuous selection for a metric projection is a continuous mapping s from X onto Y such that $s(x) \in P_Y(x)$ for every x in X.

Let X and Y be two topological spaces and $F : X \mapsto 2^Y$ a set-valued mapping. F is called lower semicontinuous if, for every open subset V of Y, the set $\{x \in X; F(x) \cap V \neq \emptyset\}$ is open on X.

Let Y be a proximinal subspace of a Banach space X. The Michael's Theorem ([4], Theorem 3.2") follows that, if P_Y is lower semicontinuous, then P_Y admits a continuous selection. However, the convers is not true (see [2]).

B. B. Panda and O. P. Kapoor[5] has shown that, if Y is a proximinal subset of a locally uniformly convex Banach space X, then there exists a dense subset G such that the restriction $P_Y|_G$ of P_Y on G admits a continuous selection.

*Received date: 2000-10-24
Foundation item: Supported by the Natural Science Foundation of Hebei
Biography: SONG Wen-hua (1956-),male, Ph.D., Professor.

In this paper, we will prove that, if Y is separable and reflexive, then there exists a subset A of the first category in X, such that $P_Y|_{X-A}$ is lower semicontinuous and admits a continuous selection.

Firstly, we introduce the following definition.

Definition *Let X be a topological space and Y a Banach space. The set-valued mapping $F : X \mapsto 2^Y$ is called weakly lower semicontinuous if, in the weak topology on Y, F is lower semicontinuous. A weakly continuous selection of F is a selection which is continuous in the weak topology on Y.*

The pupose of this paper is to discuss the properties of lower weakly semicontinuous for a metric projection and the weakly continuous selection.

In Section 2, we discuss the existence of weakly continuous selection for general set-valued mapping.

In Section 3, we apply the results of section 2 to metric projection.

2. The continuous selection of set-valued mapping

In this section, we discuss the existence of weakly continuous selection for general set-valued mapping.

Let X be a topological space, Y a Banach space and, $F : X \mapsto 2^Y$ a set valued mapping. Consider the following assmptions:

[(F_1)] There exists a $R > 0$ such that, for every x in X,

$$F(x) \subseteq \{y \in Y; \|y\| \leq R\}.$$

[(F_2)] For each x in X, $F(x)$ is closed in the norm of Y.

[(F_3)] For each x in X, $F(x)$ is convex.

Remark 2.1 If F satisfies the conditions (F_2) and (F_3), Banach-Mazur's Theorem implies that $F(x)$ is weakly closed in Y for every x in X.

In essence, the following Lemma just is the Lemma 4.1 of [4].

Lemma 2.2 *Let X be normal, paracompact topological space and Y a Banach space. $F : X \mapsto 2^Y$ is a weakly lower semicontinuous set-valued mapping. If F satisfies the conditions $(F_1) - (F_3)$, then, gievn a convex neighborhood V of the origin of Y, there exists a weakly continuous mapping $f : X \mapsto Y$ such that,*

$$f(x) \in F(x) + V, \text{ and } \|f(x)\| \leq R,$$

for every $x \in X$.

Proof By the proof of Lemma 4.1 of [4], the f may be written as

$$f(x) = \sum_{p \in P} p(x) y(p),$$

where P is a locally finite partition of unity on X which is subordinated to some open covering, and $y(p) \in F(x_p)$ for some x_p in X.

Since $p(x) \geq 0$, by (F_1), one has,

$$\|f(x)\| \leq \sum_{p \in P} p(x)\|y(p)\| \leq R(\sum_{p \in P} p(x)) = R.$$

Remark 2.3. Lemma 2.2 is as the Lemma 4.1 of [4] except for the condition $\|f(x)\| \leq R$. However, this condition is not essential to the following Theorem.

Theorem 2.4 *Let X be normal, paracompact topological space and Y a separable, reflexive Banach space. $F : X \mapsto 2^Y$ is a weakly lower semicontinuous set-valued mapping. If F satisfies the conditions (F_1) – (F_3), then f admits a weakly continuous selection.*

Proof Suppose $\{a_n\}_{n=1}^{\infty}$ being a number sequence such that $a_n > 0$ and $2^{n+1}a_{n+1} \leq a_n$. Let $\{\phi_n\}_{n=1}^{\infty}$ be a countable dense subset of the unit sphere in the dual space of Y, and

$$V_n = \{y \in Y; |\varphi_i(y)| \leq a_n,\ i = 1, 2, \cdots, n\}.$$

Then V_n is weakly open and $V_{n+1} \subseteq 2^{-(n+1)}V_n$.

By Lemma 2.2 and using the proof of (a) $\Rightarrow$ (b) in Theorem 3.2" of [4], there exists a sequence $\{f_n\}_{n=1}^{\infty}$ of weakly continuous functions from X into Y such that

$$f_{n+1}(x) \in f_n(x) + 2V_n, \quad n = 1, 2, \cdots. \tag{1}$$

$$f_n(x) \in F(x) + V_n, \quad n = 1, 2, \cdots. \tag{2}$$

$$\|f_n(x)\| \leq R, \quad n = 1, 2, \cdots. \tag{3}$$

Since Y is a reflexive Banach space, Y is weakly complete. Now, it sufficient to show that $\{f_n\}_{n=1}^{\infty}$ is uniformly Cauchy, that is, given an open neighborhood V of the origin in Y, then there exists an integer N such that,

$$f_m(x) - f_n(x) \in V, \tag{4}$$

when $n, m > N$ and $x \in X$. By (1), one has $f_{n+1}(x) - f_n(x) \in 2V_n$. Therefor,

$$\begin{aligned} f_m(x) - f_n(x) \in 2V_{m-1} + \cdots + 2V_n &\subseteq 2[(\frac{1}{2^{m-1}})V_{m-2} + \cdots + (\frac{1}{2^{n-1}})V_{n-1}] \\ &\subseteq 2(\frac{1}{2^{m-2}} + \cdots \frac{1}{2^{n-1}})V_{n-1} \subseteq V_{n-1}. \end{aligned}$$

It follows that, given an integer N, for any $m, n > N$ and every x in X ,

$$f_m(x) - f_n(x) \in V_N. \tag{5}$$

By the property of weak topology, there exist k elements $\psi_1, \cdots \psi_k$ in the nuit sphere of the dual space of Y and $\varepsilon > 0$ such that

$$V_0 = \{y \in Y;\ \ |\psi_i(y)| < \varepsilon,\ i = 1, 2, \cdots k\} \subseteq V.$$

Since $\{\varphi_n\}_{n=1}^{\infty}$ is dense in the unit sphere of the dual space of Y, there exist integers $m_1, m_2, \cdots, m_k$ such that,

$$||\varphi_{m_i} - \psi_i|| < \varepsilon/4R, \ i = 1, 2, \cdots, k.$$

Let N be an integer such that, $a_N < \varepsilon/2$ and $N \geq m_i$ for $i = 1, 2, \cdots, k$. When $n, m > N$, by (3) and (5),

$$\begin{aligned} ||\psi_i[f_m(x) - f_n(x)]|| &\leq 2R||\varphi_{m_i} - \psi_i|| + |\varphi_{m_i}[f_m(x) - f_n(x)]| \\ &\leq \varepsilon/2 + a_N < \varepsilon. \end{aligned}$$

Hence, $f_m(x) - f_n(x) \in V_0 \subseteq V$, that is, [4] is hold. □

We recall that, a subset A of a topological space X is of the first category in X, if it is a countable unin of nowhere dense subsets. Let X be a complete metric space and A a subset of the first category in X. Baire's Theorem implies that $X - A$ is dense in X.

Theorem 2.5 *Let X be a complete metric space and Y a seconed countable, locally compact Hausdorff space. The set-valued mapping $F : X \mapsto 2^Y$ is defined on X and, the set $G = \{(x, y);\ y \in F(x)\}$ is closed in $X \times Y$. Then there exists a subset A of the first category in X such that the restriction $F|_{X-A}$ of F on $X - A$ is lower semicontinuous.*

Proof By the assumption of Y, there exists a collection of open subsets $\{V_n\}_{n=1}^{\infty}$ in Y such that $\overline{V}_n$ is compact and, for each y in Y and an open neighborhood V of y there exists an integer n such that

$$y \in V_n \subseteq \overline{V}_n \subseteq V.$$

Let $F_n = \{x \in X;\ \ F(x) \cap \overline{V}_n \neq \emptyset\}$. Suppose $x_k \in F_n$ and

$$\lim_{k \to \infty} x_k = x.$$

Since $F(x_k) \cap \overline{V}_n$ is non-empty, let y_k be in it. Since $\overline{V}_n$ is compact, we may assume that

$$\lim_{k \mapsto \infty} y_k = y,$$

for some $y \in \overline{V}_n$. Then $(x_k, y_k) \mapsto (x, y)$ in $X \times Y$. Since $(x_k, y_k) \in G$ and G is closed, so $(x, y) \in G$. Hence,

$$y \in F(x) \cap \overline{V}_n, \text{ and } x \in F_n.$$

So F_n is closed and ∂F_n has no any interior points where ∂F_n denote the boundary of F_n. Let $A = \cup_{n=1}^{\infty} \partial F_n$. Then A is of the first category in X.

It remains to show that the restriction $F|_{X-A}$ of F on $X - A$ is lower semiconutinuous.

Let $x \in X - A$ and W be a open subset of Y such that $F(x) \cap W$ is non-empty. Suppose $y \in F(x) \cap W$. Then there exists an ieteger n such that

$$y \in V_n \subseteq \overline{V}_n \subseteq W. \tag{6}$$

Hence $F(x) \cap \overline{V}_n$ is non-empty, that is, $x \in F_n$. Since $x \in X - A$, $x \notin \partial F_n$, so x is an interior point of F_n. Hence there exists a neighborhood U of x such that $U \subseteq F_n$, that is,

$F(z) \cap \overline{V}_n$ is nonempty when $z \in U$. (6) implies that F is lower semicontinuous at x. $\square$

3. Application to metric projection

In this section, the results of section 2 are applied to metric projection.

Let X be a Banach space and Y a subspace of X. It is elementary that, for each $x \in X$, $P_Y(x)$ is closed and convex. If Y is reflexive, by [1], Y is proximinal. Since, for every $x \in X$,

$$\overline{\operatorname{span}\{Y \cup \{x\}\}} = Y + [x]$$

is reflexive where [x] is the subspace spanned by x.

It is well known that, if Y is a separable and reflexive Banach space, then, for each $R > 0$, on the weak topology of Y, $\{y \in Y; \|y\| \leq R\}$ is a seconed countable, locally compact Hausdorff topological space.

Theorem 3.1 *If Y is a separable, reflexive subspace of a Banach space X and P_Y is weakly lower semicontinuous, then P_Y admits a weakly continuous selection.*

Proof Let F be the restriction of P_Y on $S(X)$. Then $F : S(X) \mapsto 2^Y$ is weakly lower semicontinuous. Evidently, F satisfies the conditions $(F_1) - (F_3)$ for $R = 2$. Since $S(X)$ is normal and paracompact, using Theorem 2.5, F admits a weakly continuous selection f_0. Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \|x\| f_0(x/\|x\|) & \text{otherwise.} \end{cases}$$

It is obvious that f is a selection of P_Y. Now, it remains to show that f is weakly continuous. Given a weakly open subset V of Y, let $x_0 \in f^{-1}(V)$. We must show that there exists a neighborhood U of x_0 in X such that $f(U) \subseteq V$. Since $\|y\| \leq 2\|x\|$, for every $x \in X$ and $y \in P_Y(x)$, the proof is easy when $x_0 = 0$. Hence we may assume that $x_0 \neq 0$. Let $y_0 = f(x_0)$. Then

$$y_0/\|x_0\| = f_0(x_0/\|x_0\|).$$

Suppose $\{\varphi_i\}_{i=1}^n \subseteq Y^*$ and $\varepsilon > 0$ such that

$$y_0 + V_0 \subseteq V,$$

where

$$V_0 = \{y \in Y;\ \ \|\varphi_i(y)\| < \varepsilon,\ i = 1, 2, \cdots, n\}.$$

Let

$$V_1 = \{y \in Y;\ \|\varphi_i(y)\| < \varepsilon/3,\ i = 1, 2, \cdots, n\}.$$

Then, $3V_1 \subseteq V_0$. Since $y_0/\|x_0\| + \|x_0\|^{-1}V_1$ is open, f_0 is a weakly continuous selection and

$$f_0(x_0)/\|x_0\| \in \frac{y_0}{\|x_0\|} + \|x_0\|^{-1}V_1,$$

there exists a $\delta_0 > 0$ such that, when $\|x - \frac{x_0}{\|x_0\|}\| < \delta_0$ and $\|x\| = 1$, one has

$$f_0(x) \in y_0/\|x_0\| + \|x_0\|^{-1}V_1.$$

Then there exists a $\delta > o$ such that, when $||x - x_0|| < \delta$,

$$|||x||/||x_0||| \leq 2, \text{ and } ||\frac{x}{||x||} - \frac{x_0}{||x_0||}|| < \delta_0$$

and

$$|\frac{||x|| - ||x_0||}{||x_0||}| \mid \varphi_i(y_0) \mid < \varepsilon/3, \quad i = 1, 2, \cdots, n.$$

Hence, we have that, when $||x - x_0|| < \delta$,

$$||x|| f_0(\frac{x}{||x||}) \in \frac{||x||}{||x_0||} y_0 + \frac{||x||}{||x_0||} V_1 = \frac{||x|| - ||x_0||}{||x_0||} y_0 + y_0 + \frac{||x||}{||x_0||} V_1$$
$$\subseteq y_0 + 3V_1 \subseteq y_0 + V_0 \subseteq V.$$

Therefore, $f^{-1}(V)$ is open on the norm topology and f is a weakly continuous selection. □

Theorem 3.2 *Let Y be a separable, reflexive subspace of a Banach space X. Then there exists a subset A of the first category in X such that the restriction $P_Y|_{X-A}$ of P_Y on $X - A$ is weakly lower semicontinuous.*

Proof For each integer n, let $X_n = \{x \in X; ||x|| \leq n\}$, $S_n = \{x \in X; ||x|| = n\} = \partial X_n$ and $F_n = P_Y|_{X_n}$ the restriction of P_Y on X_n. Obviously, X_n is a complete metric space and the set-valued mapping $F_n : X_n \mapsto 2^Y$ satisfies the conditions (F_1) - (F_3) for $R = 2n$. By Theorem 2.4, there exists a subset A_n of the first category in X_n such that $F_n|_{X_n - A_n}$ is weakly lower semicontinuous. Suppos

$$A_n = \bigcup_{k=1}^{\infty} B_{n,k},$$

where $B_{n,k}$ is nowhere dence. Hence

$$A = (\cup_{n=1}^{\infty} B_{n,k}) \cup (\cup_{n=1}^{\infty} S_n) = \cup_{n=1}^{\infty} \cup_{k=1}^{\infty} (B_{n,k} \cup S_n)$$

is of the first category in X. Now, we show that $P_Y|_{X-A}$ is weakly lower semicontinuous. Let V be a weakly open subset of Y. We must prove that

$$D = \{x \in X - A;\ P_Y(x) \cap V \neq \emptyset\}$$

is open on $X - A$. Let

$$D_n = \{x \in X_n - A_n;\ P_Y(x) \cap V \neq \emptyset\}.$$

For each x in D, then $||x|| < n$ for some integer n, that is, x is in D_n. Since D_n is open on $X_n - A_n$, so there exists an open subset G of X such that

$$D_n = G \cap (X_n - A_n).$$

Let

$$U = \{x \in X;\ \|x\| < n\} \cap G \cap (X - A). \tag{7}$$

Then U is open on $X - A$. Since

$$\{x \in X;\ \|x\| < n\} \cap (X - A) \subseteq X_n - A_n,$$

(7) implies $x \in U \subseteq D$. So D is open. □

Corollary *If Y is finite-dimensional, then there exists a subset A of the first category in X such that $P_Y|_{X-A}$ is lower semicontinuous on the norm topology of X and Y.*

Proof It follows from the fact that the finite-dimensional norm space is separable and reflexive and, any two linear topologies on a finite-dimensional linear space are equivalent. □

Theorem 3.3 *Let X and Y satisfy the condition in Theorem 3.1. Then there exists a subset A of the first category in X and a mapping $f : X - A \mapsto Y$ such that*

(a) For $x \in X - A$ and $t \geq 0$, $tx \in X - A$.
(b) For $x \in X - A$ and $t \geq 0$, $f(tx) = tf(x)$.
(c) f is a weakly continuous selection of $P_Y|_{X-A}$.

Proof Theorem 2.5 implies that there exists a subset A_0 of the first category in $S(X)$ such that

$$P_Y|_{S(X)-A_0} : S(X) - A_0 \mapsto 2^Y$$

is weakly lower semicontinuous. Since $S(X) - A_0$ is a metric space, so $S(X) - A_0$ is normal and paracompact. By Theorem 2.4, $P_Y|_{S(X)-A_0}$ admits a weakly continuous selection f_0.

Let $A = \{tx;\ t > 0, x \in A_0\}$. Using the same method in the proof of Theorem 3.1, $f : X - A \mapsto Y$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ \|x\| f_0(x/\|x\|) & \text{otherwise,} \end{cases}$$

for every x in $X - A$, satisfies $(a) - (b)$. It remains to show that A is that of the first category in X. Let $A_0 = \cup_{n=1}^{\infty} B_n$, where B_n is nowhere dense in $S(X)$. Let

$$B_{n,k} = \{tx;\ t \geq k^{-1}, x \in B_n\}.$$

Then $A = \cup_{n,k=1}^{\infty} B_{n,k}$. Now, it is sufficient to show that, if $\varepsilon > 0$ and B_0 is a nowhere dence subset on $S(X)$, then

$$B = \{tx;\ t \geq \varepsilon \text{ and } x \in B_0\}$$

is nowhere dense in X. Obviously, $\overline{B} = \{tx;\ t \geq \varepsilon, x \in \overline{B}_0\}$. Suppose x_0 being an interior point of $\overline{B}$. Then there exists an open neighborhood V of x_0 such that $V \subseteq \overline{B}$. This implies that

$$(\|x_0\|^{-1} V) \cap S(X) \subseteq \overline{B}_0$$

and

$$(\|x_0\|^{-1}V) \cap S(X)$$

is an open subset on $S(X)$. This is impossible since B_0 is nowhere dence on $S(X)$. The proof is complete. □

Corollary *If Y is a finite-dimensional subspace of a Banach space X, then there exists a subset A of the first category in X and a mapping $f : X - A \mapsto Y$ such that the (a) and (b) of Theorem 3.3 are hold and*

(c') f is a continuous selection of $P_Y|_{X-A}$. □

References:

[1] SINGER I. *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces* [M]. Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[2] DEUTSCH F. *Continuous selections for metric projections some results progress* [C]. in "Approximation Theory V" (E.d., by C. K. Chui, etc.), 319 – 321, Acadmic Press, New York, 1986.

[3] DEUTSCH F, LI W, PAKK S H. *Characterizations of continuous and Lipschitz continuous metric selections in normed linear spaces* [J]. J. Approx. Theory, 1989, **58**: 297 – 314.

[4] MICHAEL E. *Continuous selections I* [J]. Ann. Math., 1956, **63** : 361 – 382.

[5] PANDA B B, KAPOOR O P. *Approximation compactness and continuity of metric projection* [J]. Bull. Austral. Math. Spc., 1974, **11**: 47–55.

度量射影的连续选择

宋 文 华

(河北经贸大学应用数学所，河北 石家庄 050091)

摘 要: 本文讨论了集值映射的弱连续选择并应用于度量射影. 设 Y 是 Banach 空间 X 的子空间且 Y 是可分的，在相差一个第一纲集的情况下，于弱拓扑下，支撑于 Y 上的度量射影是下半连续的并有连续选择.