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The Radicals of Lattice-Ordered Rings *

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Abstract: We discuss the nilpotency of nil single-sided l-ideals of lattice-ordered rings. It is shown that for many rings satisfying specific lattice-ordered ring identities all the radicals coincide.

Key words: l-radical; l-B radical; chain condition.

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1. Introduction

Johnson[5] pointed out that a nil l-ideal of a lattice-ordered ring (l-ring) may not be nilpotent. In this paper we discuss the nilpotency of nil single-sided l-ideals of l-rings. A sufficient condition for the ideal generated by a nil single-sided ideal to be nil is given.

The l-radical [1] and l-B radical [3] of an l-ring are different in general. We show that for some rings satisfying specific l-ring identities the nil radicals are equal.

In this paper, R denotes an l-ring. For nonempty subset X of R, $\langle X\rangle$ is the l-ideal of R generated by X. Suppose that A is an l-ideal of R, let $A^{(1)} = A, A^{(i)} = \langle A^{(i-1)}A\rangle, i = 2,3,\cdots$, then $\langle A^{(m)}A^{(n)}\rangle = A^{(m+n)}$, $m,n = 1,2,\cdots$. An f-ring is an l-ring in which $a \wedge b = 0$ and $c \geq 0$ imply that $ca \wedge b = ac \wedge b = 0$. An l-ring satisfying $|xy| = |x||y|$ is usually called a d-ring. The l-radical $L(R)$ (upper l-radical $U(R)$ [4]) of R is the sum of all nilpotent (nil) l-ideals of R. The l-B radical $B(R)$, the l-Q radical $Q(R)$, and the P-radical $P(R)$ are that defined in [3]. Clearly $L(R) \subseteq B(R) \subseteq U(R) \subseteq H(R) = \{x \in R \mid |x| \text{ is nilpotent }\} \subseteq \{x \in R \mid x \text{ is nilpotent}\}$.

In ring theory, the following Koethe problem is still open: Whether the ideal generated by a nil single-sided ideal is still nil, has not been solved yet. For l-rings we have the following obvious result:

Theorem 1 *(1) The l-ideal of R generated by any nil single-sided l-ideal of R is nil.*

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(2) If a nil left (right) ideal A of ring R is contained in a nil left (right) l-ideal of R, then the ring ideal generated by A is a nil ideal of ring R.

(3) The sum of some nil left (right) l-ideals of R is still a nil left (right) l-ideal of R.

Johnson[5] indicated that nil and nilpotent are different in general. Here we show some sufficient conditions for a nil single-sided l-ideals being nilpotent. Obviously the results of Birkhoff and Pierce [1]are corollaries of our conclusions.

Theorem 2 *If R satisfies descending chain condition on nil l-ideals, then every nil single-sided l-ideal A of R is nilpotent, furthermore $L(R) = B(R) = Q(R) = P(R) = U(R)$ is nilpotent.*

Proof By Theorem 1 (1) we need only to prove that A is a nil l-ideal. From the descending chain condition, there is a positive integer m such that $A^{(m)} = A^{(m+1)}$. Assume that $A^{(m)} \neq 0$, then there exists a minimal nil l-ideal J satisfying $J \subseteq A^{(m)}$ and $\langle A^{(m)} J A^{(m)} \rangle \neq 0$. Let $a \in J \cap R^+$ and $\langle A^{(m)} a A^{(m)} \rangle \neq 0$. Then $a \leq bab'$ for some $b, b' \in A^{(m)} \cap R^+$. Since A is nil, we obtain $a = 0$. This contradicts $\langle A^{(m)} a A^{(m)} \rangle \neq 0$, therefore $A^{(m)} = 0$. Thus A is nilpotent by $A^m \subseteq A^{(m)}$, which implies that $U(R) \subseteq L(R)$. So $L(R) = B(R) = Q(R) = P(R) = U(R)$ is nilpotent by [3, Theorem 3.2].

Theorem 3 *If R satisfies ascending chain condition on nilpotent l-ideals, then $B(R) = Q(R) = P(R) = L(R)$ is nilpotent.*

Proof By the ascending chain condition, R contains a maximal nilpotent l-ideal N. Let A be a nilpotent l-ideal of R, then $N = A + N$ by maximality, which implies that $L(R) \subseteq N$. Therefore $L(R) = N$ is nilpotent. It is easy to prove that $B(R) = Q(R) = P(R)$ is nilpotent via transfinite induction and [3, Theorem 3.2], and $B(R) \subseteq L(R)$. Thus $B(R) = Q(R) = P(R) = L(R)$.

Suppose that B is an l-ideal of R, let $l_B(x) = \{r \in B \mid |r||x| = 0\}$, $l(x) = \{r \in R \mid |r||x| = 0\}$, clearly $l_B(x)$ and $l(x)$ are left l-ideals of R.

Theorem 4 *Suppose that R satisfies ascending chain condition on nil l-ideals, then*

(1) if R has a nonzero nil l-ideal, then R has a nonzero nilpotent l-ideal.

(2) each nil single-sided l-ideal of R is nilpotent, and $U(R) = B(R) = Q(R) = P(R) = L(R)$ is also nilpotent.

Proof (1) Suppose that B is a nonzero nil l-ideal of R, and $\langle l_B(b) \rangle$ is maximal in $\{\langle l_B(x) \rangle \mid 0 < x \in B\}$. If $bR^+ = 0$, then $\langle b \rangle^2 = 0$. If there is $r \in R^+$ with $br > 0$, then we derive $\langle l_B(b) \rangle = \langle l_B(br) \rangle$. Since B is nil, there exists a positive integer n such that $(br)^n = 0$ and $(br)^{n-1} \neq 0$, but $(br)^{n-1} = by, 0 < y \in R$, whence $br \in \langle l_B((br)^{n-1}) \rangle = \langle l_B(b) \rangle$, and $brb = 0$, that is $bR^+b = 0$. Since every element of R is the difference of two positive elements, $bRb = 0$, so $\langle b \rangle^3 = 0$. The proof is complete. □

(2) Let N be the maximal nilpotent l-ideal of R, A a nil l-ideal of R. If A is not contained in N, then $(A + N)/N$ is a nonzero nil l-ideal of R/N, hence R/N contains a nonzero nilpotent l-ideal, which contradicts the maximality of N. It follows that $A \subseteq N$, namely nil single-sided l-ideals of R are nilpotent by Theorem 1, and $U(R) = B(R) = Q(R) = P(R) = L(R)$ is nilpotent.

Theorem 5 *(1) If R is commutative, then $L(R) = B(R) = Q(R) = P(R) = U(R) = H(R)$.*

(2) If R is an f-ring, then $L(R) = B(R) = Q(R) = P(R) = U(R) = H(R) = \{x \in R \mid x \text{ is nilpotent}\}$.

(3) If R is a d-ring, then $B(R) = Q(R) = P(R) = U(R) = H(R) = \{x \in R \mid x \text{ is nilpotent}\}$.

(4) If R is a d-ring with an identity element, then $L(R) = B(R) = Q(R) = P(R) = U(R) = H(R) = \{x \in R \mid x \text{ is nilpotent}\}$.

(5) If R is an archimedian l-ring in which the square of every element is positive, then $L(R) = B(R) = Q(R) = P(R) = U(R) = H(R)$.

Proof (1) If $x \in H(R)$, then $\langle x \rangle$ is nilpotent by commutativity, which implies that $L(R) \subseteq B(R) = Q(R) = P(R) \subseteq U(R) \subseteq H(R) \subseteq L(R)$.

(2) It follows directly from [1, Theorem 16].

(3) Let $\bar{R} = R/B(R)$. Clearly $\bar{R}$ is a d-ring. Suppose that $\bar{a} \in \bar{R}^+$ and $\bar{a}\bar{R} = \bar{0}$. Then $\bar{a} \in L(\bar{R}) \subseteq B(\bar{R}) = \{\bar{0}\}$ by [1, p45, Definition] and [3, Theorem 3.2 and 2.1]. Hence $\bar{R}$ is an f-ring by [1, p58-59, Lemma 1 and Theorem 14]. Suppose that $x \in R$ and $x^n = 0$. Then $\bar{x}^n = \bar{0}$ and $\bar{x} \in L(\bar{R}) = B(\bar{R}) = \{\bar{0}\}$, which implies $x \in B(R)$. This completes the proof.

(4) The proof is immediate from [1, p58-59, Lemma 1 and Corollary 1] and (2).

(5) Obviously by [2, Theorem 3.9].

Corollary *The ideal generated by a nil single-sided ideal of an f-ring or a d-ring is also nil.*

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格 序 环 的 根

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摘 要: 讨论格序环中诣零单侧 l- 理想的幂零性, 证明了许多满足特殊格序环等式的环的所有根相同.