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The Category of Finitely Generated Meta-Projective Left R-Modules *

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Abstract: In this paper,it is shown that for a QF ring R, the category of projective left R-modules is a category with factorization if and only if $gl.dimR \leq 1$,moreover, if $P(_RR) = P(R_R) = 0$,then the meta-Grothendieck groups obtained by left modules or by right modules are the same,up to isomorphism. It is also shown that the category of f.g. meta-pojective left R-modules is not only a category with factorization but also a category with product such that it has a small skeletal subcategory.

Key words: meta-projective module; meta-Grothendieck group; category with factorization.

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Throughout this paper, all rings are associative with $1 \neq 0$, and modules are unitary. We first recall two important concepts. One is the category with factorization, the other is the meta-Grothendieck group.

In 1972, Daniel L.Davis and Donald W.Robinson gave the definition of the category with factorization in [1]. Let C be a category. The category C is said to be a category with factorization if for any morphism $(\phi : X \to Y)$ of C, there exists an object Z, an epimorphism $\phi_1 : X \to Z$ and a monomorphism $\phi_2 : Z \to Y$ such that $\phi = \phi_2\phi_1$. It is shown that the category with factorization plays an important role in generalzed inverse theories. Suppose that R is a ring. According to [2], the meta-Grothendieck group of R is defined to be F/F_0, where F is the free Abelian group whose free generators $\langle P \rangle$ are the isomorphism classes of f.g. meta-projective R-modules P,and F_0 is the subgroup of F generated by all expressions $\langle P \rangle + \langle Q \rangle - \langle P \oplus Q \rangle$. The results of [2] showed the

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meta-Grothendieck group measured in some sense how far away an integral domain is from being a division ring, and how far away a quasi-local ring is from being a field.

Now take f.g. meta-projective left R-modules as objects and R-module homomorphisms as morplisms, then a category is obtained. The purpose of this paper is to investigate its properties of this category. We show the category of f.g. meta-projective left R-modules is not only a category with factorization but also a category with product such that it has a small skeletal subcategory. We obtain that for a QF ring R, the category of projective left R-modules is a category with factorization if and only if gl.dim$R \leq 1$, moreover, if $P({}_RR) = P(R_R) = 0$, then the meta-Grothendieck group constructed by left modules or by right modules are the same,up to isomorphism.

Lemma 1 *Let N be a submodule of R-module M. If M is meta-projective, then so is N.*

Proof By [3], Propesition 1.5, $P(N) \subset P(M)$,the result is followed immediately.

Theorem 2 *Let $\underline{\underline{mP}}(R)$ be the category of f.g. meta-projective left R-modules, and $\underline{\underline{M}}(R)$ the category of projective left R-modules.Then*

(1) $\underline{\underline{mP}}(R)$ is a category with factorication.

*(2) For a QF ring R, $\underline{\underline{M}}(R)$ is a category with factorization if and only if gl.*dim$R \leq 1$.

Proof (1) Let $(\phi : P_1 \to P_2)$ be a morphism of $\underline{\underline{mP}}(R)$. Since P_1 is f.g., Imϕ is f.g., and we have the commutative diagram:

$$\begin{array}{ccccc} P_1 & & \xrightarrow{\varphi} & & \mathrm{Im}\varphi \xrightarrow{i} P_2 \\ & \varphi \nwarrow & & \nearrow \psi & \\ & & P_1/\mathrm{Ker}\phi & & \end{array}$$

By Lemma 1, Imϕ is meta-projective, so Im$\phi \cong P_1/$Kerϕ is meta-projective,also $P_1/$Kerϕ. Hence $P_1/$Ker$\phi \in ob\underline{\underline{mP}}(R)$, and $\phi = \psi\varphi$,$\underline{\underline{mP}}(R)$ is a category with factorization.

(2) Suppose $\underline{\underline{M}}(R)$ is a category with factorization. Let P be a projective module, and N a f.g. submodule of P. Then there is an exact sequence $F \xrightarrow{\nu} N \to 0$, where F is a f.g. free module.Consider the morphism $i\nu : F \to P$, then there is a commutative diagram

$$\begin{array}{ccccc} F & & \xrightarrow{i\nu} & & P \\ & \phi_1 \nwarrow & & \nearrow \phi_2 & \\ & & P' & & \end{array}$$

where $P' \in ob\underline{\underline{M}}(R)$, ϕ_1 is right cancellative with respect to the composition of morphisms, and ϕ_2 is left cancellative with respect to composition of morphisms. We assert:

For a morphism ϕ in $\underline{\underline{M}}(R)$, ϕ is left cancellative with respect to composition of morphisms if and only if ϕ is a monomorphism as an R-homomorphism.

In fact, let $\phi : X \to Y$ and there be $x_1, x_2 \in X$ such that $x_1 \neq x_2$ but $\phi(x_1) = \phi(x_2)$.Then $0 \neq x_1 - x_2 \in X$,$\phi(x_1 - x_2) = 0$. The homomorphism $\Psi : R \to R(x_1 - x_2) \to X, r \mapsto r(x_1 - x_2)$ implies $\Psi \in \mathrm{mor}(R, X)$ and $\Psi \neq 0$. So $\phi\Psi = \phi 0$, the assertion follows.

Since Imϕ_1 is a f.g. submodule of P', there is the following exact sequence

$$0 \to \mathrm{Im}\phi_1 \to P' \to P'/\mathrm{Im}\phi_1 \to 0.$$

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Note that R is self-injective,we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Im}\phi_1 & \to & P' & \to & P'/\mathrm{Im}\phi_1 & \to & 0 \\ & & \mu_1 \downarrow & & \downarrow \mu_2 & & \downarrow \mu & & \\ 0 & \to & (\mathrm{Im}\phi_1)^{**} & \to & (P')^{**} & \to & (P'/\mathrm{Im}\phi_1)^{**} & \to & 0 \end{array}$$

$P' \in ob\underline{\underline{M}}(R)$ induces μ_2 is an R-monomorphism, R is a QF ring implies that μ_1 is an R-isomorphism. By the Five Lemma,we obtain μ is an R-monomorphism. On the other hand, the left exactity of the functor $\hom(-,R)$ guarantees that $(P'/Im\phi_1)^{**}$ can be injected into a free module F'. Thus $P'/\mathrm{Im}\phi_1$ can also be injected into a free module F'. If $\mathrm{Im}\phi_1 \neq P'$, then $\varphi' : P' \xrightarrow{\varphi} P'/\mathrm{Im}\phi_1 \xrightarrow{i} F'$ and $0 : P' \to F'$ are two different R-homomorphisms. But $\varphi'\phi_1 = 0\phi_1$, contradicts to the right cacellativity of ϕ_1. Now $N \cong P'$,$N \in ob\underline{\underline{M}}(R)$. So R is semi-hereditary, wgl.dim$R \leq 1$. But R is Notherian, so gl.dim$R$ = wgl.dim$R \leq 1$.

Conversely, gl.dim$R \leq 1$ is equivalent to the fact that submodules of projective modules are still projective.If $\phi \in \mathrm{mor}(P_1, P_2)$, then $\mathrm{Im}\phi$ is projective.So the following diagram

$$\begin{array}{ccccc} P_1 & & \xrightarrow{\Phi} & & P_2 \\ & \varphi \searrow & & \nearrow i & \\ & & \mathrm{Im}\Phi & & \end{array}$$

shows that $\underline{\underline{M}}(R)$ is a category with factorization.The proof is completed. □

Proposition 3 *Let $\underline{\underline{m}}(R)$ be the category of meta-projective modules.(g_1, P_1, f_1), (g_2, P_2, f_2) are two standard factorizations of morphism $\phi : P \to Q$. Then*

(1) There is a pull back (D, h_1, h_2) on f_1 and f_2 in $\underline{\underline{m}}(R)$.

(2) For the pull back (D, h_1, h_2), there exists an unique morphism $\Psi : P_1 \to D$ such that $g_1 = h_1\Psi$, $g_2 = h_2\Psi$.

(3) $(h_1, P_1, f_1), (h_2, P_2, f_2)$ are still standard factorizatons of some morphism from D to Q.

Proof (1) In the category of R-modules, let $M = \{(a_1, a_2)|f_1(a_1) = f_2(a_2), a_i \in P_i, i = 1, 2\}$. Then $(M, p_1|_M, p_2|_M)$ is a pull back of f_1 and f_2, where p_i is the projection from $P_1 \times P_2$ to P_i. From[4], Theorem 4, M is meta-projective, so (1) is followed.

(2) and (3), by[5], Theorem 3, immediately.

Theorem 4 *Let $\underline{\underline{mP}}(R)$ be the category of f.g. meta-projective modules. Then $(\underline{\underline{mP}}(R), \oplus)$ is a category with product such that $\underline{\underline{mP}}(R)$ has a small skeletal subcategory.*

Proof Let $M_1, M_2, M_3 \in ob\underline{\underline{mP}}(R)$. Denote $\theta : M_1 \oplus (M_2 \oplus M_3) \to (M_1 \oplus M_2) \oplus M_3, m_1 \oplus (m_2 \oplus m_3) \mapsto (m_1 \oplus m_2) \oplus m_3$, and denote $\phi : M_1 \oplus M_2 \to M_2 \oplus M_1, m_1 \oplus m_2 \mapsto m_2 \oplus m_1$. By [6] Lemma 1, it is easy to see that $(\underline{\underline{mP}}(R), \oplus)$ is a category with product. If the cardinal of the underlying set of R is λ, then R^n has cardinal λ^n. For any f.g. R-module M, by the exact sequence $0 \to \mathrm{Ker}\varphi \to \overset{n}{\oplus} R \xrightarrow{\varphi} M \to 0$. We have $M \cong \overset{n}{\oplus} R/Ker\varphi$. But the number of submodules of $\overset{n}{\oplus} R$ cannot exceed 2^{λ^n}, it is clear that the category of f.g. R-modules has a small skeletal category. Naturally $\underline{\underline{mP}}(R)$ has a small skctetat category. The proof is completed. □

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Usig the language of category, we can define the meta-Gothendieck group of R, as the abelian group given by the following generators and relations: we take one generator $[P]_s$ for each isomorphism class of objects P in $\underline{\underline{mP}}(R)$ and one relation $[P]_s \oplus [Q]_s = [P \oplus Q]_s$ for each pair $P, Q \in ob\underline{\underline{mP}}(R)$.

Corollary 5 *Let R be a QF ring such that $P(_RR) = P(R_R) = 0$. Then $sK_0\underline{\underline{mP}}_l(R) = sK_0\underline{\underline{mP}}_r(R)$.*

Proof Suppose $M \in ob\underline{\underline{mP}}_l(R)$, $\overset{n}{\oplus} R \to M \to 0$ induces $0 \to M^* \to (\overset{n}{\oplus} R)^* \cong \overset{n}{\oplus} R$. Thus M^* is meta-projective.So $M \to M^*$ is a product-preserving contravariant functor $\underline{\underline{mP}}_l(R) \overset{*}{\longrightarrow} \underline{\underline{mP}}_r(R)$ and so induces a homomorphism $sK_0\underline{\underline{mP}}_l(R) \to sK_0\underline{\underline{mP}}_r(R)$ by the rule $[M]_s \to [M^*]_s$. Similarly $\underline{\underline{mP}}_r(R) \overset{*}{\longrightarrow} \underline{\underline{mP}}_l(R)$ induces $sK_0^* : sK_0\underline{\underline{mP}}_r(R) \to sK_0\underline{\underline{mP}}_l(R)$. The composite $(sK_0^*)\overline{(sK_0^*)} = sK_0^{**} : sK_0\underline{\underline{mP}}_l(R) \to sK_0\underline{\underline{mP}}_l(R)$ is given by $[M]_s \to [M^{**}]_s$. Note that R is a QF ring,so every f.g. R-module are reflexive,i.e., $M \cong M^{**}$. Thus $[M]_s = [M^{**}]_s$ and sK_0^{**} is the identity homomorphism. Similarly on $sK_0\underline{\underline{mP}}_r(R)$. Thus $sK_0^* : sK_0\underline{\underline{mP}}_l(R) \to sK_0\underline{\underline{mP}}_r(R)$ is an isomophism. Then the result is followed.

Given a QF ring R such that $P(_RR) = P(R_R) = 0$,the Corollary above shows that it the meta-Grothendieck group constructed by left modules or by right modules are the same, up to isomorphism.

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有限生成亚投射模范畴

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摘 要: 对一个 QF 环 R, 本文证明: 其投射左 R 模范畴是因式分解范畴当且仅当 gl.dim$R \leq 1$. 进一步, 若 $P(_RR) = P(R_R) = 0$, 则其通过左模而得到的亚 Grothendieck 群与其通过右模而得到的亚 Grothendieck 群在同构意义下是一样的. 还证明了有限生成亚投射左 R- 模范畴不仅是一个因式分解范畴而且是一个带积的具有小的骨架子范畴的范畴.

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