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On Cobordism Classes of Fiber Bundles *

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Abstract: In this paper we deal with some properties of cobordism classes of fiber bundles over the real projective space.

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1. Introduction

Let MO_m be the unoriented cobordism group. Then $MO_* = \sum MO_m$ is a Z_2-polynomial algebra with a single generator in each dimension m which is not of the form $2^u - 1$ ([1]). If a cobordism class $\alpha_m \in MO_m$ can be expressed as a sum of products of lower dimensional cobordism classes, then α_m is decomposable. Otherwise it is indecomposable. The indecomposable classes can be chosen as generators of MO_*.

Given a cobordism class $\alpha_m \in MO_m$, one says that α_m fibers over N^n with fiber F^{m-n} if there is a differentiable fibering of closed manifolds

$$F^{m-n} \hookrightarrow M^m \xrightarrow{\pi} N^n,$$

where M^m belongs to the class α_m.

For $N^n = RP(n)$, the real projective space, Stong proved

Proposition 1 *There are indecomposable classes $\alpha_m \in MO_m$ fibered over $RP(j)$ for all $j \leq n$ where*

(a) $m = 2q \geq 4, n = 2q - 2,$

and

(b) $m = 2^p(2q+1) - 1, p, q > 0, n = 2^{p+1}q - 2.$

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The following question was raised by R.E.Stong[2]: Are these the largest values of n for which there is an indecomposabe class α_m fibered over $RP(n)$ with $m > n$? For m even, this is true. In this paper we answer the question for $m = 2^k + 1, 11, 13$.

Let Z_2 denote the integers mod 2 , $W_i(*)$ Stiefel-Whitney class and $W_i[*]$ Stiefel-Whitney number. Throughout this paper we work with Z_2 coeffcients. Binomial cofficients are $\binom{m}{n} = m!/n!(m-n)!$.

2. Main results and proofs

First, we mention the following well known elementary facts. Let $RP(n_1, n_2, \cdots, n_l)$ be the projective space bundle of $\lambda_1 \oplus \lambda_2 \oplus \cdots \oplus \lambda_l$ over $RP(n_1) \times RP(n_2) \times \cdots \times RP(n_l)$, where λ_i is the pullback of the canonical line bundle over the ith factor. Then the cobordism class $[RP(n_1, n_2, \cdots, n_l)] (l > 1)$ is indecomposable in MO_* if and only if

$$\binom{n+l-2}{n_1} + \binom{n+l-2}{n_2} + \cdots + \binom{n+l-2}{n_l} \equiv 1 \mod 2,$$

where $n = n_1 + n_2 + \cdots + n_l$. The manifold $RP(n_1, n_2, \cdots, n_l)$ has dimension $n_1 + n_2 + \cdots + n_l + l - 1$. If $n_{i+1} = n_{i+2} = \cdots = n_l = 0$, then $RP(n_1, n_2, \cdots, n_l)$ will sometimes be written as $RP(n_1, \cdots, n_i; l)$.

Let $F^{m-n} \overset{i}{\hookrightarrow} M^m \overset{\pi}{\longrightarrow} N^n$ be a fibering of closed manifolds. From [3] we have

$$TM^m \approx \pi^*(TN^n) \oplus \overline{T}F^{m-n},$$

where TM^m is the tangent bundle to M^m, TN^n the tangent bundle to N^n and $\overline{T}F^{m-n}$ the bundle of tangent vectors parallel to the fibre F^{m-n}.The total Stiefel-Whitney class

$$W(M^m) = \pi^*(W(N^n))W(\overline{T}F^{m-n}).$$

Thus for $N^n = RP(n)$,

$$W(M^m) = (1 + \pi^*(d))^{n+1} W(\overline{T}F^{m-n}),$$

where $d \in H^1(RP(n)), d^{n+1} = 0$.

Theorem 1 *For $m = 2^k + 1$, if a cobordism class α_m fibers over $RP(2^k - 1)$, then $\alpha_m = 0$.*

Proof We take generators x_l of MO_* as follows. For odd $l = 2^p(2q+1) - 1$ $(p > 0, q > 0), x_l = RP(2^p; l - 2^p + 1)$, and so the Stiefel-Whitney class $W_1(x_l) = 0$; For even $l = 2p, x_l = RP(2p)$. A direct computation shows that x_l are indecomposabe. If $\alpha_m = [M^m]$ fibers over $RP(2^k - 1)$, then there is a fibering

$$F^2 \hookrightarrow M^m \overset{\pi}{\longrightarrow} RP(2^k - 1).$$

The total Stiefel-Whitney class $W(M^m) = (1 + \pi^*(d))^{2^k} W(\overline{T}F^2) = 1 + W_1 + W_2$, $d \in H^1(RP(2^k - 1))$ and $d^{2^k} = 0$. From [1,Theorem 17.1], to prove the theorem we need only to show that $W_1^iW_2^j[M^m] = 0, i+2j = m$. Let $[M^m] = \sum x_{l_1}x_{l_2}\cdots x_{l_s}, l_1+l_2+\cdots+l_s = m$. We assert that in each monomial $x_{l_1}x_{l_2}\cdots x_{l_s}$ there exists some x_{l_p} such that l_p is odd because $m = 2^k+1$ is odd. Consideration of $W_1(x_{l_p}) = 0$ reveals $W_1^iW_2^j[x_{l_1}x_{l_2}\cdots x_{l_s}] = 0$. Therefore $W_1^iW_2^j[M^m] = 0$. The proof is completed.

Theorem 2 *If a cobordism class $\alpha_{11} \in MO_{11}$ fibers over $RP(7)$, then $\alpha_{11} = 0$.*

Proof If $\alpha_{11} = [M^{11}]$ fibers over $RP(7)$, then there is a fibering

$$F^4 \hookrightarrow M^{11} \xrightarrow{\pi} RP(7).$$

The total Stiefel-Whitney class $W(M^{11}) = (1+\pi^*(d))^8W(\overline{T}F^4)$. Since $d^8 = 0$, it follows that $W_i(M^{11}) = 0, i \geq 5$. Thus $W_2^3W_5[\alpha_{11}] = W_1^2W_2W_7[\alpha_{11}] = W_1W_4W_6[\alpha_{11}] = W_1^4W_2W_5[\alpha_{11}] = 0$. From [4] there exists no indecomposable class in MO_{11} which fibers over the 4-dimensional sphere S^4, but [5,Proposition 2] indicates that α_{11} fibers over S^4. Hence α_{11} is decomposable, that is, $S_{11}[\alpha_{11}] = 0$, where $S_{11} \in H^{11}(M^{11})$ is the polynomial in the Stiefel-Whitney classes corresponding to the symmetric function $t_1^{11}+t_2^{11}+\cdots+t_{11}^{11}$.

Take the indecomposable classes $x_2 = RP(2), x_4 = RP(1,1,0), x_5 = RP(2,0,0,0)$, $x_6 = RP(3,0,0,0), x_9 = RP(2;8), x_{11} = RP(4;8)$. A computation shows

	x_{11}	x_2x_9	x_5x_6	$x_2x_4x_5$	$x_2^3x_5$
S_{11}	1	0	0	0	0
$W_2^3W_5$	1	0	1	1	1
$W_1^2W_2W_7$	0	1	0	0	1
$W_1W_4W_6$	0	0	1	1	0
$W_1^4W_2W_5$	0	0	0	1	0

Since this matrix is nonsingular, it is not difficult to see $\alpha_{11} = 0$.

Theorem 3 *If a cobordism class $\alpha_{13} \in MO_{13}$ fibers over $RP(11)$, then $\alpha_{13} = 0$.*

Proof If $\alpha_{13} = [M^{13}]$ fibers over $RP(11)$, then there is a fibering

$$F^2 \hookrightarrow M^{13} \xrightarrow{\pi} RP(11)$$

The total Stiefel-Whitney class

$$W(M^{13}) = (1+\pi^*(d))^{12}W(\overline{T}F^2) = (1+\pi^*(d))^{12}(1+v_1+v_2), d^{12} = 0.$$

A computation shows that $W_3(M^{13}) = W_7(M^{13}) = (W_4W_6+W_2W_4^2)(M^{13}) = 0$. Hence all Stiefel-Whitney numbers of M^n divisible by W_3, W_7 and $W_4W_6+W_2W_4^2$ are zero.

	x_{13}	x_2x_{11}	$x_2^4x_5$	$x_4^2x_5$	$x_2x_5x_6$	$x_2^2x_9$	x_4x_9	$x_2^2x_4x_5$	x_5x_8
W_3W_{10}	1	*	*	*	*	*	*	*	*
$W_1^2W_2W_3^3$	0	1	*	*	*	*	*	*	*
W_6W_7	0	0	1	*	*	*	*	*	*
$W_2^5W_3$	0	0	0	1	*	*	*	*	*
$W_1^3W_4W_6+$ $W_1^3W_2W_4^2$	0	0	0	0	1	*	*	*	*
$W_2^2W_3W_6$	0	0	0	0	0	1	*	*	*
$W_1^4W_3W_6$	0	0	0	0	0	0	1	*	*
$W_1^4W_2^3W_3$	0	0	0	0	0	0	0	1	*
$W_1^8W_2W_3$	0	0	0	0	0	0	0	0	1

Take the indecomposable classes $x_2, x_4, x_5, x_6, x_9, x_{11}$ as in Theorem 2 and $x_8 = RP(3,3,0)$, $x_{13} = RP(2;12)$. Characteristic numbers are given by

Since this matrix is nonsingular, $\alpha_{13} = 0$. We complete the proof.

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上协边类纤维丛表示的若干性质

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摘 要: 本文研究了底空间为实射影空间时上协边类纤维丛表示的若干性质.