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L^p-Bounds for a Marcinkiewicz Integral on Product Domains *

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Abstract: In this paper, we proved the L^p-boundedness of the Marcinkiewicz integral $\mu_\Omega(f)$ on product domains $R^n \times R^m$,where $\Omega \in L(\log^+ L)^{2\beta}(S^{n-1} \times S^{m-1})(\ \beta > 1)$.

Key words: Marcinkiewicz integral; rough kernel; product domains; square functions.

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1. Introduction

Recently, many authors have studied the L^p-boundedness for the singular integral operator T on product domains $R^n \times R^m$ defined by

$$T(f)(x,y) = p.v. \iint_{R^n \times R^m} \frac{\Omega(x-u, y-v)}{|x-u|^n |y-v|^m} f(u,v) \mathrm{d}u \mathrm{d}v$$

for Ω in different function spaces on $S^{n-1} \times S^{m-1}$. For example , in [1] R.Fefferman proved that T is a bounded operator on $L^p(R^n \times R^m)$ if Ω satisfies some regularity conditions. In [2], J.Duoandikotxea proved the L^p-boundedness of $T(f)$ for $\Omega \in L^q(S^{n-1} \times S^{m-1})(q > 1)$. For some $\beta > 1$ and $\Omega \in L(\log^+ L)^{2\beta}(S^{n-1} \times S^{m-1})$, Hu[5] proved the L^p-boundedness of $T(f)$.

In this paper we shall consider the L^p-bounds for the Marcinkiewicz integral operator $\mu(f)$ on product domains, defined by

$$\mu(f)(x,y) = \left(\int_0^{+\infty} \int_0^{+\infty} |F_{t,s}(x,y)|^2 \frac{\mathrm{d}t \mathrm{d}s}{t^3 s^3}\right)^{\frac{1}{2}}, \tag{1}$$

where

$$F_{t,s}(x,y) = \iint_{\substack{|u| \le t \\ |v| \le s}} \frac{h(|u|,|v|)\Omega(u,v)}{|u|^{n-1}|v|^{m-1}} f(x-u, y-v) \mathrm{d}u \mathrm{d}v, \tag{2}$$

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and $\Omega \in L^1(S^{n-1} \times S^{m-1})$ which satisfies the following conditions,

$$\Omega(tx', sy') = \Omega(x', y'), \forall t, s \in R^+, \tag{3}$$

$$\int_{S^{n-1}} \Omega(x', y') \mathrm{d}x' = 0, \forall y' \in S^{m-1}, \tag{4}$$

$$\int_{S^{n-1}} \Omega(x', y') \mathrm{d}y' = 0, \forall y' \in S^{n-1}. \tag{5}$$

In [4], Yong Ding proved that $\Omega \in L(\log^+ L)^2(S^{n-1} \times S^{m-1})$ is sufficient for the L^2-boundedness of $\mu(f)$. Inspired by the work[3], we shall not deal with its original operator here but with its equivalent one $\mu_\Omega(f)$ defined by

$$\mu_\Omega(f)(x,y) = (\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |F_{t,s}^{\Omega}(x,y)|^2 \frac{\mathrm{d}t\mathrm{d}s}{2^{2t}2^{2s}})^{\frac{1}{2}}, \tag{6}$$

$$F_{t,s}^{\Omega}(x,y) = \iint_{\substack{|u|\le 2^t \\ |v| \le 2^s}} \frac{h(|u|,|v|)\Omega(u,v)}{|u|^{n-1}|v|^{m-1}} f(x-u, y-v) \mathrm{d}u\mathrm{d}v. \tag{7}$$

Now we state the main result of this paper.

Theorem 1 *Let Ω be as above and $h \in L^\infty(R^+ \times R^+)$. Suppose that Ω belongs to the space $L(\log^+ L)^{2\beta}(S^{n-1} \times S^{m-1})$ for some $\beta > 1$. Then the Marcinkiewicz integral defined by (6) is L^p-bounded on $L^p(R^n \times R^m)$ for $p \in (\frac{2\beta}{2\beta-1}, 2\beta)$.*

2. Basic lemmas

We begin with some notations and lemmas, which will be used in the proof of our theorem. For $t, s \in R$, we define measures $\{\sigma_{t,s}\}$,$\{\lambda_{t,s}\}$ by

$$\widehat{\sigma_{t,s}}(\xi_1, \xi_2) = \iint_{\substack{|u|\le 2^t \\ |v| \le 2^s}} h(|u|,|v|)\Omega(u,v)|u|^{1-n}|v|^{1-m} e^{-i(u\cdot\xi_1 + v\cdot\xi_2)} \mathrm{d}u\mathrm{d}v/2^{t+s}, \tag{8}$$

$$\widehat{\lambda_{t,s}}(\xi_1, \xi_2) = \iint_{\substack{|u|\le 2^t \\ |v| \le 2^s}} |h(|u|,|v|)\Omega(u,v)||u|^{1-n}|v|^{1-m} e^{-i(u\cdot\xi_1 + v\cdot\xi_2)} \mathrm{d}u\mathrm{d}v/2^{t+s}. \tag{9}$$

Also we define the maximal operator $\sigma^*(f)$ by

$$\sigma^*(f)(x,y) = \sup{}_{t,s\in R} |\lambda_{t,s} * f(x,y)|.$$

Then it's easy to see that

$$2^{-t-s} F_{t,s}^{\Omega}(x,y) = \sigma_{t,s} * f(x,y). \tag{10}$$

Lemma 1

$$|\widehat{\lambda}_{t,s}(\xi_1,\xi_2)| \le \iint_{\substack{|u|\le 2^t \\ |v| \le 2^s}} |h(|u|,|v|)|u|^{1-n}|v|^{1-m}\Omega(u,v)| \mathrm{d}t\mathrm{d}s/2^{t+s}$$
$$\le c||\Omega||_1||h||_\infty, \tag{11}$$

$$||\sigma^*(f)||_p \le c||\Omega||_1||f||_p \text{ for } 1 < p < +\infty, \tag{12}$$

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where c is independent of $t, s \in R$.

Lemma 2 *Let* Ω_0 *be a function defined on* $S^{n-1} \times S^{m-1}$ *and belong to the space* $L^\infty(S^{n-1} \times S^{m-1})$, *then for* $\xi_1 \in R^n, \xi_2 \in R^m, \xi_2 \cdot \xi_1 \neq 0$

$$|\widehat{\sigma_{t,s}}(\xi_1, \xi_2)| \leq c||\Omega_0||_\infty(|2^t\xi_1||2^s\xi_2|)^{-\varepsilon}$$

for constants c and $\varepsilon > 0$ *which are independent of* Ω_0, t, s.

Proof For fixed $\xi_1 \in R^n, \xi_2 \in R^m$, we write

$$I_{r,\rho}(\xi_1, \xi_2) = \iint_{S^{n-1} \times S^{m-1}} \Omega_0(u', v')e^{-iru'\cdot\xi_1 - i\rho v'\cdot\xi_2}\mathrm{d}u'\mathrm{d}v',$$

then we have

$$\begin{aligned}
|\widehat{\sigma_{t,s}}(\xi_1, \xi_2)| =& |\iint_{S^{n-1} \times S^{m-1}} \Omega_0(u', v') \int_0^{2^t}\int_0^{2^s} e^{-iru'\cdot\xi_1 - i\rho v'\cdot\xi_2} h(r, \rho)\frac{\mathrm{d}r\mathrm{d}\rho}{2^{t+s}}\mathrm{d}u'dv'| \\
=& \int_0^{2^t}\int_0^{2^s} I_{r,\rho}(\xi_1, \xi_2)h(r, \rho)\frac{\mathrm{d}r\mathrm{d}\rho}{2^{t+s}} \\
\leq& c||h||_\infty(\int_0^{2^t}\int_0^{2^s} |I_{r,\rho}(\xi_1, \xi_2)|^2\frac{\mathrm{d}r\mathrm{d}\rho}{2^{t+s}})^{\frac{1}{2}} \\
\leq& c||h||_\infty(\int_0^1\int_0^1 |I_{2^t r, 2^s\rho}(\xi_1, \xi_2)|^2\mathrm{d}r\mathrm{d}\rho)^{\frac{1}{2}},
\end{aligned}$$

write

$$\begin{aligned}
|I_{r,\rho}(\xi_1, \xi_2)|^2 =& \iint_{(S^{n-1} \times S^{m-1})^2} \Omega_0(w', z')\Omega_0(u', v')e^{-ir(u'-w')\cdot\xi_1 - i\rho(v'-z')\cdot\xi_2} \times \\
& e^{-iru'\cdot\xi_1 - i\rho v'\cdot\xi_2}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z',
\end{aligned}$$

thus we obtain for any $0< \varepsilon < 1$

$$\begin{aligned}
|\widehat{\sigma_{t,s}}(\xi_1, \xi_2)|^2 \leq& c||h||_\infty^2 \iint_{(S^{n-1} \times S^{m-1})^2} \Omega_0(w', z')\Omega_0(u', v') \int_0^1\int_0^1 |I_{2^t r, 2^s\rho}(\xi_1, \xi_2)|^2\mathrm{d}r\mathrm{d}\rho \\
\leq& c||h||_\infty^2 \iint_{(S^{n-1} \times S^{m-1})^2} |\Omega_0(w', z')\Omega_0(u', v')||2^t\xi_1|^{-\varepsilon}|2^s\xi_2|^{-\varepsilon}\mathrm{d}z' \times \\
& |(u' - w')\cdot\xi_1'|^{-\varepsilon}|(v' - z')\cdot\xi_2'|^{-\varepsilon}\mathrm{d}u'\mathrm{d}v'\mathrm{d}w'\mathrm{d}z' \\
\leq& c||h||_\infty^2||\Omega_0||_\infty|2^t\xi_1|^{-\varepsilon}|2^s\xi_2|^{-\varepsilon}(\iint_{S^{m-1} \times S^{m-1}} |(v' - z')\cdot\xi_2'|^{-\varepsilon}\mathrm{d}v'\mathrm{d}z') \times \\
& (\iint_{S^{n-1} \times S^{n-1}} |(w' - u')\cdot\xi_1'|^{-\varepsilon}\mathrm{d}w'\mathrm{d}u') \\
\leq& c||h||_\infty^2||\Omega_0||_\infty|2^t\xi_1|^{-\varepsilon}|2^s\xi_2|^{-\varepsilon}.
\end{aligned}$$

Lemma 3 *Let* $\overline{\Omega}$ *be a function on* $S^{n-1}\times S^{m-1}$ *for* $n \geq 2, m \geq 2$ *and* $\overline{\Omega} \in L(\log^+ L)^\gamma(S^{n-1}\times S^{m-1})$ *for some* $\gamma > 1$. *Then for any positive integer k, there exists a function* $\overline{\Omega}_k$ *defined on* $S^{n-1} \times S^{m-1}$ *satisfying* ,

$$||\overline{\Omega}_k||_\infty \leq c2^k, ||\overline{\Omega} - \overline{\Omega}_k||_1 \leq ck^{-\gamma}.$$

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This result is contained in the paper[5].

Lemma 4 *Let $t,s\in R,\Omega\in L(\log^+L)^{2\beta}$ and satisfies (3)-(5), then*

(i) $\widehat{\sigma_{t,s}}(\xi_1,\xi_2)|\le c|2^t\xi_1||2^s\xi_2|$ *if* $2^t\xi_1|\le 2,|2^s\xi_2|\le 2$;

(ii) $\widehat{\sigma_{t,s}}(\xi_1,\xi_2)|\le c(\log|2^t\xi_1|)^{-2\beta}|2^s\xi_2|$ *if* $|2^t\xi_1|\ge 2,|2^s\xi_2|\le 2$;

(iii) $\widehat{\sigma_{t,s}}(\xi_1,\xi_2)|\le c(\log|2^s\xi_2|)^{-2\beta}|2^t\xi_1|$ *if* $|2^t\xi_1|\le 2,|2^s\xi_2|\ge 2$;

(iv) $\widehat{\sigma_{t,s}}(\xi_1,\xi_2)|\le c(\log|2^s\xi_2||2^t\xi_1|)^{-2\beta}$ *if* $|2^t\xi_1|\ge 2,|2^s\xi_2|\ge 2$.

Proof We discuss the following four conditions.

Case(i) It's easy to be obtained once we observe the cancellations of Ω.

Case(ii) We write

$$\begin{aligned}|\hat{\sigma}_{t,s}(\xi_1,\xi_2)| =& |\iint_{S^{n-1}\times S^{m-1}}\Omega(u',v')\int_0^1\int_0^1 e^{-i2^tu'\cdot\xi_1 r}(e^{-i2^s v'\cdot\xi_2 s}-1)h(r,\rho)\mathrm{d}r\mathrm{d}\rho\mathrm{d}u'\mathrm{d}v'|\\ &\le ||h||_\infty|2^s\xi_2|\int_0^1|\int_{S^{n-1}}\Omega(u',v')e^{-i2^tu'\cdot\xi_1 r}\mathrm{d}u'|\mathrm{d}v'\mathrm{d}r\\ &\le c||h||_\infty|2^s\xi_2||\int_0^1|\int_{S^{n-1}}(\Omega(u',v')-\Omega_k(u',v'))e^{-i2^tu'\cdot\xi_1 r}\mathrm{d}u'|\mathrm{d}v'\mathrm{d}r+\\ &\quad c||h||_\infty|2^s\xi_2||\int_0^1|\int_{S^{n-1}}\Omega_k(u',v')e^{-i2^tu'\cdot\xi_1 r}\mathrm{d}u'|\mathrm{d}v'\mathrm{d}r\\ &\le c||h||_\infty|2^s\xi_2|||\Omega(u',v')-\Omega_k(u',v')||_1+I.\end{aligned}$$

As the lemma2 above, we can obtain ,

$$|I|\le c||h||_\infty|2^s\xi_2||2^t\xi_1|^{-k}||\Omega_k||_\infty \text{ for any } 0<k<\frac{1}{2}.$$

If $|2^t\xi_1|\ge 2$, denote l_k the positive integer such that

$$2^{l_k}\le|2^t\xi_1|^{\frac{k}{2}}\le 2^{l_k+1},$$

while Lemma3 tells us that we may choose Ω_k satisfying

$$||\Omega_k||_\infty\le c2^{l_k},\ ||\Omega-\Omega_k||_1\le cl_k^{-2\beta}.$$

This in turn implies that

$$|\hat{\sigma}_{t,s}(\xi_1,\xi_2)|\le c||h||_\infty|2^s\xi_2|(||\Omega-\Omega_k||_1+|2^t\xi_2|^{-k}||\Omega_k||_\infty)\le c||h||_\infty|2^s\xi_2|(\log{}_2|2^t\xi_1|)^{-2\beta}.$$

Case (iii) The proof is exactly similar to the case (ii).

Case (iv) Similarly by lemma3 we have

$$|\hat{\sigma}_{t,s}(\xi_1,\xi_2)|\le c||h||_\infty|2^s\xi_2|^{-\epsilon}|2^t\xi_1|^{-\epsilon}||\Omega_{k_0}||_\infty+c||h||_\infty||\Omega-\Omega_{k_0}||_1.$$

If $|2^t\xi_1|\ge 2$ and $|2^s\xi_2|\ge 2$, we choose l such that$2^l\le(|2^t\xi_1||2^s\xi_2|)^{\frac{\epsilon}{2}}\le 2^{l+1}$ while Lemma3 tells us that for the positive integer l, there exists a function Ω_l defined on $S^{n-1}\times S^{m-1}$,such that

$$||\Omega_l||_\infty\le c2^l\ ||\Omega-\Omega_l||_1\le cl^{-2\beta}.$$

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Then

$$|\widehat{\sigma}_{t,s}(\xi_1,\xi_2)| \le c||h||_\infty |2^s\xi_2|^{-\frac{\varepsilon}{2}}|2^t\xi_1|^{-\frac{\varepsilon}{2}} + c'||h||_\infty[\log_2|2^t\xi_1||2^s\xi_2|]^{-2\beta}.$$

This yields the desired estimates.

Now, we take two functions $\Phi_1 \in C_0^\infty(R^n)$, $\Phi_2 \in C_0^\infty(R^m)$, radial, supp$(\Phi_i) \subset \{1 \le |x_i| \le 2\}$ and $\Phi_i \ge 0, \int_0^{+\infty} \Phi_i(t)\frac{\mathrm{d}t}{t} = 1$ where $i = 1,2$. Let $\widehat{\psi}_{t,s}(\xi_1,\xi_2) = \Phi_1(2^t\xi_1)\Phi_2(2^s\xi_2)$ and $\widehat{\widetilde{\psi}}_{t,s}(\xi_1,\xi_2) = \Phi_1(t\xi_1)\Phi_2(s\xi_2)$.

Lemma 5 *For $f, \widehat{f} \in L^1(R^n \times R^m)$, there holds*

$$f(x,y) = \lim_{\substack{\varepsilon_1,\varepsilon_2\to 0+\\ \delta_1,\delta_2\to+\infty}} \int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_1}^{\delta_2} \widetilde{\psi}_{t,s} * f(x,y)\frac{\mathrm{d}t\mathrm{d}s}{ts} \tag{13}$$

for all $(x,y) \in R^n \times R^m$ provided f is the continuous representative of the equivalence class determined by $f \in L^1(R^n \times R^m)$.

Proof If we let $f_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(x,y) = \int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_1}^{\delta_2} \widetilde{\psi}_{t,s} * f(x,y)\frac{\mathrm{d}t\mathrm{d}s}{ts}$, then by Fubini's theorem and Young's inequality,

$$||f_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}||_1 \le \log(\frac{\delta_1}{\varepsilon_1})\log(\frac{\delta_2}{\varepsilon_2})||\Phi_1||_1||\Phi_2||_1||f||_1.$$

Hence $\widehat{f}_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2) = \widehat{f}(\xi_1,\xi_2)\int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_1}^{\delta_2} \Phi_1(t\xi_1)\Phi_2(s\xi_2)\frac{\mathrm{d}t\mathrm{d}s}{ts}$. It's easy to see that $|\widehat{f}_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2)| \le |\widehat{f}(\xi_1,\xi_2)|$ which means

$$\widehat{f}_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2) \in L^1(R^n \times R^m),$$

then by the Fubini inversion theorem,

$$f(x,y) - f_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(x,y) = (\widehat{f} - \widehat{f}_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2})^\vee(x,y) \text{ for every } (x,y) \in R^n \times R^m.$$

Putting $h_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2;x,y) = e^{i(x\cdot\xi_1+y\cdot\xi_2)}\widehat{f}(\xi_1,\xi_2)[1 - \int_{\varepsilon_1}^{\delta_1}\int_{\varepsilon_1}^{\delta_2} \Phi_1(t\xi_1)\Phi_2(s\xi_2)\frac{\mathrm{d}t\mathrm{d}s}{ts}]$, so that $f(x,y) - f_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(x,y) = c_n \iint_{R^n\times R^m} h_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2;x,y)\mathrm{d}\xi_1\mathrm{d}\xi_2$ implies that

$$\lim_{\substack{\varepsilon_1,\varepsilon_2\to 0+\\ \delta_1,\delta_2\to+\infty}} h_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2;x,y) = 0 \text{ for almost everywhere } \xi = (\xi_1,\xi_2).$$

Since $|h_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(\xi_1,\xi_2;x,y)| \le |\widehat{f}(\xi_1,\xi_2)|$, the Lebesgue dominated convergence theorem yields

$$\lim_{\substack{\varepsilon_1,\varepsilon_2\to 0+\\ \delta_1,\delta_2\to+\infty}} f(x,y) - f_{\varepsilon_1,\varepsilon_2}^{\delta_1,\delta_2}(x,y) = 0 \text{ for all } (x,y)$$

which completes the lemma.

For ψ defined as above , we define the corresponding square g-function on the product domains $R^n \times R^m$,

$$g_\psi(f)(x,y) = (\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |\psi_{t,s} * f(x,y)|^2\mathrm{d}t\mathrm{d}s)^{\frac{1}{2}}.$$

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Then we can immediately prove the following lemma, once we make a slight modification on the paper [1]$P_{123-125}$.

Lemma 6 *For ψ defined as above, there holds*

$$||g_{\psi}(f)||_p \leq c||f||_p, \tag{14}$$

where c is independent of the function $f \in L^p(R^n \times R^m)$.

Now we write, for $f \in S(R^n \times R^m)$,

$$\begin{aligned}\mu_{\Omega}(f)(x,y) &= \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\sigma_{t,s} * f(x,y)|^2 \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2}} \\ &= \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sigma_{t,s} * \psi_{t+t_1,s+s_1} * f(x,y)\mathrm{d}t_1\mathrm{d}s_1\right|^2 \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2}} \\ &\leq \left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\sigma_{t,s} * \psi_{t+t_1,s+s_1} * f(x,y)|^2 \mathrm{d}t\mathrm{d}s\right)^{\frac{1}{2}}\mathrm{d}t_1\mathrm{d}s_1\right. \\ &\triangleq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I_{t_1,s_1}(f)(x,y)\mathrm{d}t_1\mathrm{d}s_1,\end{aligned} \tag{15}$$

where the second inequality follows from lemma5 and the third inequality from the Minkowski's inequality.

Lemma 7 *For $\psi, \{\sigma_{t,s}\}$ and $I_{t_1,s_1}(f)$ defined as above, then $\forall p_0 \in (1,+\infty)$, we have*

$$||I_{t_1,s_1}(f)||_p \leq c_{n,p_0}||f||_p. \tag{16}$$

The proof is exactly similar to the proof of lemma1 in the paper[2,P_{189}], once we apply lemma6 and lemma1.

3. Proof of Theorem

According to (15), that is to say

$$\mu_{\Omega}(f)(x,y) \leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} I_{t_1,s_1}(f)(x,y)\mathrm{d}t_1\mathrm{d}s_1.$$

By the Minkowski's inequality, we have $\forall p \in (1,+\infty)$

$$\begin{aligned}||\mu_{\Omega}(f)(x,y)||_p &\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}||I_{t_1,s_1}(f)(x,y)||_p \mathrm{d}t_1\mathrm{d}s_1 \\ &\leq \Big(\int_{-1}^{1}\mathrm{d}s_1\int_{-\infty}^{-1} + \int_{-1}^{1}\mathrm{d}s_1\int_{1}^{+\infty} + \int_{-1}^{1}\mathrm{d}s_1\int_{-1}^{1}\Big) + \\ &\quad \int_{-\infty}^{-1}\mathrm{d}s_1\int_{-\infty}^{-1} + \int_{-\infty}^{-1}\mathrm{d}s_1\int_{1}^{+\infty} + \int_{-\infty}^{-1}\mathrm{d}s_1\int_{-1}^{1} + \\ &\quad \int_{1}^{+\infty}\mathrm{d}s_1\int_{-\infty}^{-1} + \int_{1}^{+\infty}\mathrm{d}s_1\int_{1}^{+\infty} + \int_{1}^{+\infty}\mathrm{d}s_1\int_{-1}^{1}\Big)||I_{t_1,s_1}(f)(x,y)||_p \mathrm{d}t_1.\end{aligned}$$

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By lemma7, we immediately get ,

$$\int_{-1}^{1} \mathrm{d}s_1 \int_{-1}^{1} ||I_{t_1,s_1}(f)(x,y)||_p \mathrm{d}t_1 \le c||f||_p.$$

For the sake of simplicity, we only discuss the following four conditions and the others can be obtained similarly.

(a) If $s_1, t_1 \ge 1$, then

$$\begin{aligned} ||I_{t_1,s_1}(f)(x,y)||_2^2 \le & \iint_{(R^1)^2} \iint_{R^n \times R^m} |\widehat{f}(\xi_1,\xi_2)|^2 |\Phi_1(2^{t+t_1}\xi_1)\Phi_2(2^{s+s_1}\xi_2)|^2 \times \\ & |\widehat{\sigma}(\xi_1,\xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}t \mathrm{d}s \\ \le & c \iint_{(R^1)^2} \iint_{E_{t,s}} |\widehat{f}(\xi_1,\xi_2)|^2 |2^t\xi_1|^2 |2^s\xi_2|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}t \mathrm{d}s, \end{aligned}$$

where $E_{t,s} = \{(\xi_1,\xi_2) \in R^n \times R^m : 2^{-1-t_1} \le |\xi_1 2^t| \le 2^{-t_1}, 2^{-1-s_1} \le |\xi_2 2^s| \le 2^{-s_1}\}$. Hence

$$||I_{t_1,s_1}(f)(x,y)||_2^2 \le c2^{-2t_1}2^{-2s_1} \iint_{(R^1)^2} \iint_{E_{t,s}} |\widehat{f}(\xi_1,\xi_2)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}t \mathrm{d}s \le c2^{-2t_1}2^{-2s_1}||f||_2^2. \quad (17)$$

Interpolating beween (16) and (17) , we can easily have, $\forall p \in (1,+\infty)$

$$\int_{1}^{+\infty} \mathrm{d}s_1 \int_{1}^{+\infty} ||I_{t_1,s_1}(f)(x,y)||_p \mathrm{d}t_1 \le c_p||f||_p.$$

(b) If $s_1 \ge 1$ and $-1 \le t_1 \le 1$, we have

$$||I_{t_1,s_1}(f)(x,y)||_2^2 \le c \iint_{(R^1)^2} \iint_{E_{t,s}} |\widehat{f}(\xi_1,\xi_2)|^2 |2^s\xi_2|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}t \mathrm{d}s \le c2^{-2s_1}||f||_2^2. \quad (18)$$

Interpolating between (16) and (18), we easily get $\forall p \in (1,+\infty)$

$$\int_{1}^{+\infty} \mathrm{d}s_1 \int_{-1}^{1} ||I_{t,s}(f)(x,y)||_p \mathrm{d}t_1 \le c_p||f||_p.$$

(c) If $s_1 \ge 1$ and $t_1 < -1$, we have

$$\begin{aligned} ||I_{t_1,s_1}(f)(x,y)||_2^2 \le & c \iint_{(R^1)^2} \iint_{E_{t,s}} |\widehat{f}(\xi_1,\xi_2)|^2 |2^s\xi_2|^2 [\log_2 |2^t\xi_1|]^{-4\beta} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}t \mathrm{d}s \\ \le & c|t|^{-4\beta} 2^{-2s_1} ||f||_2^2. \end{aligned}$$

Then $\forall p \in (\frac{4\beta}{4\beta-1}, 4\beta)$ there exists $\theta_p > 1$ and $\varepsilon_p > 0$ such that

$$||I_{t_1,s_1}(f)||_p \le c_p|t|^{-\theta_p} 2^{-\varepsilon_p s_1} ||f||_p.$$

Thus

$$\int_{1}^{+\infty} \mathrm{d}s_1 \int_{-\infty}^{-1} ||I_{t_1,s_1}(f)(x,y)||_p \mathrm{d}t_1 \le c_{p,\beta}||f||_p.$$

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(d) If $s_1 < -1$ and $t_1 < -1$, then

$$\begin{aligned}||I_{t_1,s_1}(f)(x,y)||_2^2 \leq & c\iint_{(R^1)^2}\iint_{E_{t,s}} |\hat{f}(\xi_1,\xi_2)|^2[\log_2|2^s\xi_2||2^t\xi_1|]^{-4\beta}d\xi_1 d\xi_2 dt ds \\ \leq & c|t_1+s_1|^{-4\beta}||f||_2^2. \end{aligned} \tag{19}$$

Interpolating between (16) and (19), we see that , for $p \in (\frac{2\beta}{2\beta-1}, 2\beta)$, there exists $\theta_p > 1$ such that

$$||I_{t_1,s_1}(f)||_p \leq c|t_1+s_1|^{2\theta_p}||f||_p$$

which yields for all $p \in (\frac{2\beta}{2\beta-1}, 2\beta)$

$$\int_{-\infty}^{-1} ds_1 \int_{-\infty}^{-1} ||I_{t_1,s_1}(f)(x,y)||_p dt_1 \leq c_{p,\beta}||f||_p.$$

Now Theorem1 is proved.

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乘积空间上 Marcinkiewicz 积分的 L^p 有界性

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摘 要: 本文证明了乘积空间 $R^n \times R^m$ 上 Marcinkiewicz 积分 $\mu_\Omega(f)$ 的 L^p 有界性, 其中 $\Omega \in L(\log^+ L)^{2\beta}(S^{n-1} \times S^{m-1}), \beta > 1$.

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